AN INTRODUCTION TO THE DOCTRINE OF FLUXIONS.

By JOHN ROWE.

The SECOND EDITION,
With ADDITIONS and ALTERATIONS.

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MDCCLXII.

TO

WILLIAM DAVY Esq;

SERJEANT at LAW,

THIS

T R A C T

IS INSCRIBED,

BY

His most Obedient

Humble Servant,

The AUTHOR.

THE

PREFACE.

OF all the Mathematical Sciences, the *Doctrine of Fluxions* is the most extensive and sublime. By this, many Difficulties, unsurmountable by any other known Method, are solved with uncommon Expedition, Elegance, and Ease.

THE Lengths of curve Lines, the Areas of plane curvilineal Spaces, and the Surfaces and Solidities of convex Bodies; though in some few Figures they may be determined

by

by *the Ancients* Method of Exhauſtions, *Cavalerius*'s Geometry of Indiviſibles, or Dr. *Wallis*'s Arithmetic of Infinites *; yet, the Fluxional Method is the only one by which they can be obtained in General.

FLUXIONS are likewiſe a General Method for determining the Maxima and Minima of Quantities; drawing Tangents to Curves, finding their Points of Inflection, and the Radii of Curvature. By this Method alſo, are derived Theorems for calculating of Sines, Tangents, and Secants; and for making Logarithms of any Form whatever. Nor is it confined here; but extends to the inveſtigating a great Variety, not only of other Mathematical, but alſo of Philoſophical Problems.

THE *Method of Fluxions* was firſt invented about the Year 1665, by that Prince of Mathematicians and Philoſophers the late Sir *Iſaac Newton*, then Mr. *Newton* about 23 Years old: But, as he gave no public Speci-

* The *Geometry of Indiviſibles*, was firſt printed in the Year 1635; and the *Arithmetic of Infinites*, in the Year 1656.

Specimens thereof till the Appearance of his excellent *Principia* in the Year 1687, the celebrated *German* Mr. *Godfrey William Leibnitz*, to whom it was communicated in the Year 1676, applied it in the *Acta Eruditorum* printed at *Leipsic* in 1684, to a few Maxima and Minima, and Tangents to Curves, and therein claimed the Invention himself.——Their *Notations* indeed are different *; and, as Quantities are, by both, in Effect, considered as produced by continual Increase, after the same Manner as Space is described by a Body in Motion, instead of the *Velocity* with which a Quantity flows at any Point or Term of the Time in which it is supposed to be generated, called by Sir *Isaac* a *Fluxion*, *Leibnitz* takes the *Increment*, or little Part generated in an indefinitely small Portion of Time, and calls it a *Differential*.——But, as entering farther into the Controversy is foreign to my Design, those who

* *Leibnitz* denotes the *Differential* of any variable Quantity x, by dx; its *Second Differential*, by ddx. And Sir *Isaac*, for its *Fluxion*, writes $\dot{x}$; for its *Second Fluxion*, $\ddot{x}$: but in his *Principia*, flowing Quantities are expressed by the capital Letters X, Y, Z, and their *Fluxions* by the small Letters x, y, z, respectively.

who are inclined to ſee more thereof, I muſt refer to the *Commercium Epiſtolicum*, publiſhed by Order of the *Royal Society*, or to Mr. *Raphſon's Hiſtory of Fluxions*; wherein, Sir *Iſaac* is fully proved, the Original Inventor of this noble and moſt extenſive Method.

SEVERAL excellent Treatiſes have been wrote on the Subject; but, as they appear not calculated to introduce the young Beginner into this abſtruſe and difficult Science; ſo, in Order to his underſtanding them, a plain and eaſy *Introduction* ſeems to be Neceſſary: And for that End, the *Firſt Impreſſion* of the following Tract, publiſhed in the Year 1751, was wholly deſigned.

IN this *ſecond Edition*, I have made many conſiderable *Alterations* and *Additions*; by which it is rendered much more plain, full, and correct than the former.——It is divided into *Two Parts*: The *Firſt* treats of the *Direct Method of Fluxions*; in which, from the generated Quantity or *Fluent* being given, we find the *Fluxion*: And the *Second*, of the *Inverſe Method*; wherein, from the *Fluxion* being

being known, we find the *Fluent* *. And, at the End, is added, an *Appendix*, containing miſcellaneous Queſtions with their Anſwers; which could not well be inſerted with the Examples for Illuſtration, and to ſome of which there was Occaſion to refer.

THOSE Things in the former Edition occaſioned by Hints communicated by an ingenious and learned Friend, are in this ſecond ſtill retained; but, thoſe Others in which another Hand was concerned, are here omitted.

IN Order to a thorough Underſtanding of this *Introduction*, it is requiſite that the Learner be well acquainted with Arithmetic, Algebra, Geometry, Plane-Trigonometry, Conic-Sections, and the Nature of Logarithms. But, ſince many geometrical and algebraical Treatiſes, give not the Deſcriptions, and from thence the Deduction of the Properties, of ſome Curves to be found in

a the

* The *Direct Method of Fluxions*, as delivered by *Leibnitz*, is, by Foreigners, called *Calculus Differentialis*; and the *Inverſe Method*, *Calculus Integralis*.

the following Sheets; nor the Method of reducing Quantities into Infinite Series, and the new Way of noting their Powers and Roots, neceſſary to be uſed in Fluxional Tracts; therefore, theſe Deficiencies are herein ſupplied, though they do not immediately relate to the Buſineſs in Hand.

JOHN ROWE.

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CONTENTS.

PART I.

PART

PART II.

APPENDIX.

AN

AN INTRODUCTION TO THE DOCTRINE OF FLUXIONS.

PART I.

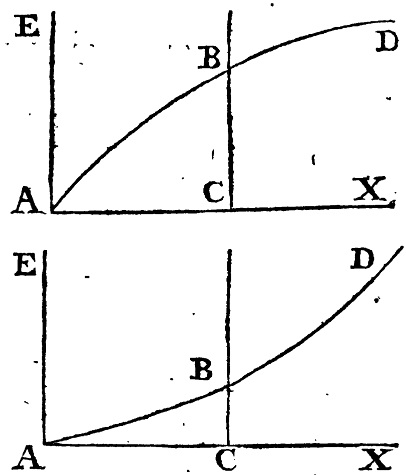
CHAP. I.

Of the Principles *of* Fluxions, *and of* the New Notation *in* Algebra.

IN this Doctrine, Quantities are ſuppoſed to be generated by continual Increaſe, after the Manner of a Space which a Body in Motion deſcribes. Thus, a Line is ſuppoſed to be generated by a Point in Motion, a Superficies by a Line, and a Solid by a Superficies,

DEFINITION I.

1. Fluxion, is the *Velocity* with which a Quantity, at any Point or Term of it, flows or increaſes.



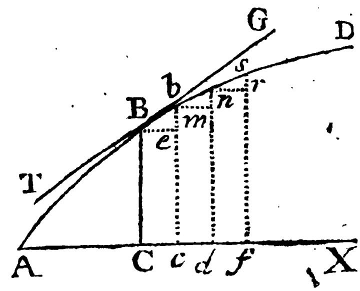
Thus, if we ſuppoſe the indefinite right Line A E to move with a parallel Motion along the Axis A X, or ſo as always to be parallel to its firſt Situation.; and, at the ſame Time, ſuppoſe a Point to move from A along the ſaid Line AE, ſo as to generate or always to be in the Curve AD: Then, the *Velocity* with which the End or Point A, of the Line AE, arrives at any Point C, or, which is the ſame, the *Velocity* with which the Axis flows or is generated at any particular Point C, is the *Fluxion* of the Abſciſs AC at that Point: The *Velocity* with which the Point moves along the Line AE at any Point B, is the *Fluxion* of the Ordinate at that

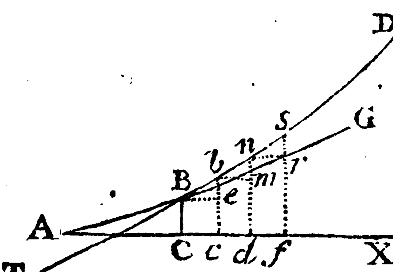
that Point: And, the *Velocity* with which the Point generates or moves along the Curve at any particular Point B, is the *Fluxion* of the Curve at that Point. So likewife, the *Velocity* with which the curvilineal Space A C B flows, or is generated by the Line A E, at any Term C B, is the *Fluxion* of the faid curvilineal Space at that Term.

2. Now, if the *Velocity* with which any Quantity flows, or is generated, be at every Point or Term the fame; that is, if it be neither accelerated nor retarded; the *Fluxion* of it will likewife be at every Point or Term the fame: But, if this Velocity be continually increafed or diminifhed; then, there will be a certain Degree of *Velocity*, or *Fluxion*, peculiar to every Point or Term of the Thing defcribed; And, the *Velocity* wherewith the given Velocity, at any Point or Term, is either accelerated or retarded, is called the *Fluxion of the Fluxion*, or the *Second Fluxion*. And, again, if this Acceleration or Retardation be not uniform, but is increafed with different Degrees of Proportion at different Points or Terms; then, the *Velocity*, or *Swiftnefs*, of this Acceleration or Retardation, is called the *Third Fluxion*. And fo on.

DEFINITION II.

3. Increment, is the indefinitely fmal Increafe of a Quantity, generated in an indefinitely fmall Particle of Time.





Thus, if we fuppofe *bc* indefinitely near and parallel to the Ordinate B C, and B*e* parallel to the Abfcifs AC; then C c, or its equal B*e*, will be the *Increment* of the Abfcifs A C; *e b* the *Increment* of the Ordinate CB; B *b* the *Increment* of the Curve A B; and C B *b c* the *Increment* of the curvilineal Space ACB.

4. Now, if *n d* be fuppofed indefinitely near and parallel to *b c*, and *b m* equal and parallel to B *e* or *c d*; then, the *Difference* between *e b* and *m n*, is called the *Increment of the Increment*, or the *Second Increment*; i. e. the *Increment* of *e b*, or *Second Increment* of C B. And, again, if *s f* be fuppofed indefinitely

finitely near and parallel to *n d*, and *n r* equal and parallel to *b m* or *d f*; then, the *Difference* between the *Second Increment* and the Difference of *m n* and *r s*, is called the *Third Increment* of C B. And ſo on.

5. *Note*, WHEN a Quantity, inſtead of increaſing, is continually diminiſhed; then, the indefinitely ſmall Particles by which it is leſſened, are not called *Increments*, but *Decrements*: And both *Increments* and *Decrements* are ſometimes called *Moments*.

6. Now, if we ſuppoſe the Abſciſs A C to flow on with an uniform Motion, or equal Parts of it to be deſcribed in equal Times; its *Increment* will *accurately* expreſs, or be *exactly* as its *Fluxion*; ſince Velocity is always as the Space uniformly deſcribed in a given Time: And, if the Curve B *b* did *exactly* coincide with the Tangent or right Line T B G; it is evident, that then, the *Increments* B *b* and *b e*, would likewiſe be deſcribed with uniform Motions, and with the ſame Degrees of Velocity with which the Curve and Ordinate reſpectively flow at the Point B; that is, the *Increments* B *b* and *b e*, would then be *accurately* as the *Fluxions* of the Curve and Ordinate at the Point B. But, ſince not two Points of the Curve are coincident with the

the Tangent, and therefore the Velocities with which the *Increments* are generated, are continually varying in every Point; ſo therefore, the *Increments* and *Fluxions* are not in an *exact* Proportion to each other; or, the *Increments* do not *accurately* meaſure the *Velocities* or *Fluxions* with which they begin to be generated. However, as the Point *b* is continually nearer to a Coincidence with the Tangent B G, the nearer it approaches the Point of Contact B; ſo, therefore, if we conceive the Ordinate $\overline{c}b$, to move back, till it coincides with C B; then, the very firſt Moment before its Coincidence, the Curve B *b* and right Line B G, will be infinitely, or rather indefinitely near to a Coincidence with each other; and conſequently, in that Caſe, the *Increments* B *b* and *b e* will come indefinitely near to meaſure the *Fluxions* of the Curve and Ordinate, or the *Velocities* with which they reſpectively flow at the Point B. Or, becauſe the Particles of Time in which any *Increments* are generated, are ſuppoſed to be indefinitely ſmall, and conſequently the Acceleration or Retardation of the Velocity with which they are generated muſt be ſo too; therefore, they are indefinitely near in Proportion to the *Fluxions* of the Quantities

of

of which they are *Increments*: But, when the Ratio from that of Equality, is but indefinitely ſmall, or leſs than can be aſſigned, it may be conſider'd as equal*; and therefore, the *Increments* may be taken as *Proportional to*, or (becauſe Velocity is expreſſed by the Space uniformly deſcribed in a given Time) *for* the *Fluxions* in all Operations; and, on the contrary, the *Fluxions* for the *Increments*.

7. *Note*. THOSE Quantities which are ſuppoſed to flow, or to be generated by continual Increaſe, are called *Fluents*, and *variable* or *flowing* Quantities: and thoſe which admit of no Variation, (as the Parameter of a Parabola, the Diameter of a Circle, &c.) are called *permanent*, *given*, and *invariable* Quantities.——The *Beginning* of the Alphabet, *viz.* a, b, c, d, &c, is uſed to expreſs *invariable* Quantities; and the *End*, *viz.* z, y, x, &c. *variable* or *flowing* Quantities. And the *Fluxions* of thoſe variables, *viz.* z, y, x, &c. are reſpectively denoted by $\dot{z}$, $\dot{y}$, $\dot{x}$, &c. and their *Second Fluxions*, or the *Fluxions* of

* This was, in effect, allowed by the ancient Geometricians, *Euclid*, *Archimedes*, &c.

of $\dot{z}$, $\dot{y}$, $\dot{x}$, *&c.* by $\ddot{z}$, $\ddot{y}$, $\ddot{x}$, *&c.* their *Third Fluxions*, or the *Fuxions* of $\ddot{z}$, $\ddot{y}$, $\ddot{x}$, *&c.* by $\dddot{z}$, $\dddot{y}$, $\dddot{x}$, *&c.* And so on. Also, the *Moment*, *Increment*, or *Decrement*, of z, is denoted by z', of y by y', of x by x', *&c.* And, the *Second Moments*, *Increments*, or *Decrements* of z, y, x, *&c.* or the *Moments*, *Increments* or *Decrements* of z', y', x', *&c.* by z'', y'', x'', *&c*, And so on.——The *Fluxions* and *Increments*, *&c.* of *invariable* Quantities, *viz.* of a, b, c, d, e, *&c.* are $=o$.

8. *Note.* Those *Fluents* which are supposed to be generated in the same Time, or in equal Times, are called *contemporary Fluents*; and the *Fluxions* of these *contemporary Fluents*, are called *contemporary Fluxions.* Now, it is evident, if two, or more, of these *contemporary Fluents*, are always equal, or in any certain constant Ratio to each other, that their *contemporary Fluxions* will likewise be equal, or in the same Proportion: And that, on the contrary, if two, or more, of these *contemporary Fluxions* are always equal, or in any certain constant Ratio to each other, their *contemporary Fluents* will likewise be equal, or in the same Proportion.

WE come now to find the *Fluxions* of *Fluents*, or the *Velocities* with which *flowing Quantities* increaſe or decreaſe at any Terms aſſigned; the Buſineſs of which, is called the *Direct Method of Fluxions.* But, before we proceed, it may, perhaps, be neceſſary to premiſe a few Things concerning

THE NEW METHOD OF NOTATION IN ALGEBRA *.

9. THE Index of the Power to which any Quantity is to be raiſed, is here placed as the Numerator of a Fraction, whoſe Denominator is the Index of the Root to be extracted.

Thus, $\sqrt[3]{xx}$ is here expreſſed by $\overline{x}|^{\frac{2}{3}}$ or $x^{\frac{2}{3}}$; and $\sqrt{xxx}$ by $x^{\frac{3}{2}}$; and $\sqrt{x}$ by $x^{\frac{1}{2}}$; and $\sqrt{a+x}$ by $\overline{a+x}|^{\frac{1}{2}}$: Alſo $\frac{1}{\sqrt[3]{xx}}$ is expreſſed by $\frac{1}{x^{\frac{2}{3}}}$ or $x^{-\frac{2}{3}}$; and $\frac{1}{\sqrt{a+x}}$ by $\frac{1}{\overline{a+x}|^{\frac{1}{2}}}$ or $\overline{a+x}|^{-\frac{1}{2}}$; and $\frac{1}{x^{3}}$ by x^{-3}.

NOW, in any geometrical Progreſſion, whoſe firſt Term is *Unity*, or 1; if we take an arithmetical Progreſſion, whoſe firſt Term

* This was introduced by the celebrated Dr. *Wallis.*

is o, and whose second Term (or common Difference) is the Index of the Quantity in the second Term of the geometrical Progression; then, the Number in any Term of the arithmetical Progression, will be the Index of the Quantity in the corresponding Term of the geometrical Progression; And therefore, the arithmetical Mean between the Numbers in any two Terms of arithmetical Progression, will be the Index of the geometrical Mean between the Quantities in the two corresponding Terms of the geometrical Progression.

Thus, in the geometrical Progression $1 . x^{\frac{1}{2}} . x^{1} . x^{\frac{3}{2}} . x^{2} . x^{\frac{5}{2}} . x^{3}$. *&c*, where the common Multiplier is $x^{\frac{1}{2}}$; if we take the arithmetical Progression $0 . \frac{1}{2} . 1 . \frac{3}{2} . 2 . \frac{5}{2} . 3$. *&c.* wherein the common Difference, or Quantity added, is $\frac{1}{2}$; the Number in either of the Terms of the arithmetical Progression, is the Index of the Quantity in the corresponding Term of the geometrical Progression: And the arithmetical Mean between the Numbers in any two Terms of the arithmetical Progression, is the Index of the geometrical Mean between the Quantities in the two corresponding Terms of the geometrical Progression. For Instance, $\frac{3}{2}$, the 4th

Term

Term of the arithmetical Progression, is the Index of $x^{\frac{3}{2}}$, the 4th Term of the geometrical Progression: And, the arithmetical Mean between $\frac{1}{2}$ and $\frac{5}{2}$ (the 2d and 6th Terms of the arithmetical Progression) which is $\frac{3}{2}$, is the Index of the geometrical Mean between $x^{\frac{1}{2}}$ and $x^{\frac{5}{2}}$ (the same two Terms of the geometrical Progression) which is $x^{\frac{3}{2}}$. — So likewise, in the descending geometrical Progression, or Series, $1.\frac{1}{x^{\frac{1}{2}}}.\frac{1}{x}.\frac{1}{x^{\frac{3}{2}}}.\frac{1}{x^2}.\frac{1}{x^{\frac{5}{2}}}.\frac{1}{x^3}.$ &c. i. e. $1.x^{-\frac{1}{2}}.x^{-1}.x^{-\frac{3}{2}}.x^{-2}.x^{-\frac{5}{2}}.x^{-3}.$ &c. where the common Divisor is $x^{\frac{1}{2}}$; if we take the arithmetical Series $0.-\frac{1}{2}.-1.-\frac{3}{2}.-2.-\frac{5}{2}.-3.$ &c. the Number in any Term of the arithmetical Series, will be the Index of the Quantity in the corresponding Term of the geometrical Series: Thus, $-\frac{3}{2}$, the 4th Term of the arithmetical Series, is the Index of $x^{-\frac{3}{2}}$, the same Term of the geometrical Series. And, the arithmetical Mean between any two Terms in the arithmetical Series, is the Index of the geometrical Mean between the same two Terms in the geometrical Series: As, for Instance, the arithmetical Mean between $-\frac{1}{2}$ and $-\frac{5}{2}$ (the 2d and 6th Terms of the arithmetical Series) which is

$-\frac{3}{2}$, is the Index of the geometrical Mean between $x^{-\frac{1}{2}}$ and $x^{-\frac{5}{2}}$ (the 2d and 6th Terms of the geometrical Series) which is $x^{-\frac{3}{2}}$.

Hence, we may obſerve, that, in the Exponents of Powers, Addition has the Effect of Multiplication on the reſpective Roots; and Multiplication of Involution: And, *e contra*, Subtraction of Diviſion; and Diviſion of Evolution: Or that, in a Word, Exponents of Powers, are entirely *Logarithmical* with regard to their Roots.

So that $x^2 \times x^{\frac{3}{2}}$ is $= x^{2+\frac{3}{2}} = x^{\frac{7}{2}}$; $\frac{x^1}{x^1}$ is $= x^{1-1} = x^0 = 1$; $\frac{x^{\frac{2}{3}}}{x^{\frac{1}{2}}}$ is $= x^{\frac{2}{3}-\frac{1}{2}} = x^{\frac{1}{6}}$; $\overline{x^3}|^{2}$ is $= x^{3\times 2} = x^{6}$; $\overline{x^4}|^{\frac{1}{2}}$ is $= x^{\frac{4}{2}} = x^{2}$. And, Univerſally, $x^m \times x^n$ is $= x^{m+n}$; $\frac{x^m}{x^n}$ is $= x^{m-n}$; $\overline{x^m}|^{n}$ is $= x^{mn}$; $\overline{x^m}|^{\frac{1}{n}}$ is $= x^{\frac{m}{n}}$; and $\frac{\overline{x^m}|^{a}}{x^{\frac{a}{n}}}$ is $= x^{ma-\frac{a}{n}} = x^{\frac{mn-1}{n}a}$: Where *Note*, *a*, *m*, *n*, repreſent any affirmative, or negative Whole-numbers, or Fractions whatever.

If what has been ſaid, be duly conſidered, no Difficulty in this *New Notation* will occur; the Knowledge of which, is abſolutely neceſſary, in Order to the well underſtanding the following Chapters.

CHAP.

CHAP. II.

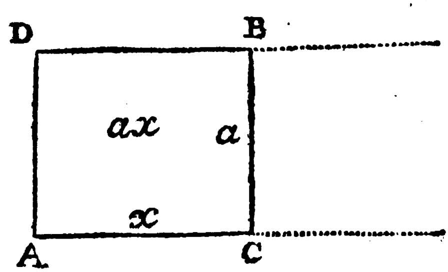
Of finding the FLUXION *of a given* FLUENT.

RULE I.

10. *To find the* Fluxion *of a* simple Fluent; *or, of that wherein there is but* one *variable Letter.*

MARK the variable Letter, or flowing Term, with a Dot over it.

Thus, the *Fluxion* of ax is $= a\dot{x}$. For, if we suppose the variable Rectangle A B,



to be generated by the given Line C B, setting out from the Situation A D, and moving along with a parallel Motion between the parallel and indefinite Sides AC and DB; it is evident, the Velocity with which the Rectangle flows, is equal to the generating Line C B drawn into the Velocity with which the Point C generates, or moves along the Line A C: that is, the *Fluxion* of the Rectangle A B, is equal to the invariable Line

Line C B drawn into the *Fluxion* of the flowing or variable Line A C. Therefore, if A D or C B $= a$, and A C or D B $= x$, then the *Fluxion* of the Rectangle $a x$ will be $= a \times \dot{x} = a \dot{x}$.

Rule II.

11. *To find the* Fluxion *of the Product of two or more Quantities drawn into each other.*

Multiply the *Fluxion* of each Quantity separately, by the other, or the Product of the rest of the Quantities: And the Sum of these Products will be the *Fluxion* required.

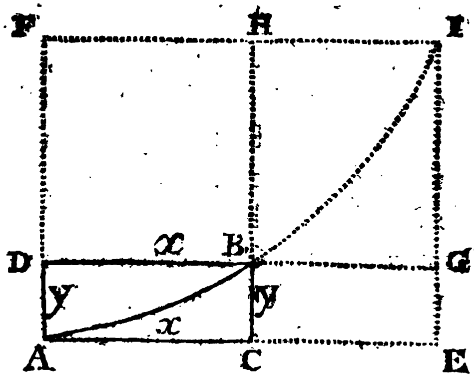
Thus, the *Fluxion* of $x y$ is $= \dot{x} y + x \dot{y}$; the *Fluxion* of $a x y$ is $= o x y + a \dot{x} y + a x \dot{y}$, i. e. $= a \dot{x} y + a x \dot{y}$; the *Fluxion* of $x y z$ is $= \dot{x} y z + x \dot{y} z + x y \dot{z}$; and the *Fluxion* of $a x y z$ is $= a \dot{x} y z + a x \dot{y} z + a x y \dot{z}$.

1°. That the *Fluxion* of $x y$ is $= \dot{x} y + x \dot{y}$, may be thus demonstrated.

Suppose a Line, coincident with the indefinite right Line A F, to move with a parallel Motion along the indefinite right Line A E; and, at the same Time, another, coincident with the Line A E, to move with a parallel Motion along the Line A F; and that the said two Lines move with such different Degrees of Velocity, that their Points

of

of Intersection may always be in the Curve A I. Thro' *any* Point B of the Curve,



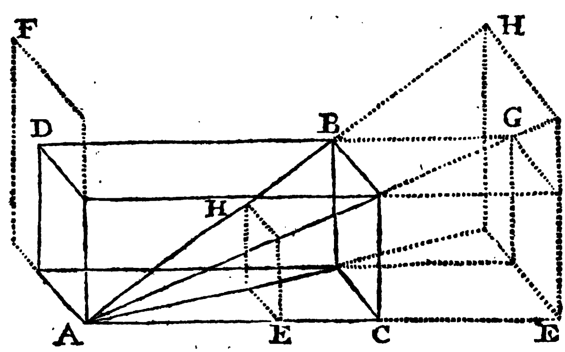
draw C H parallel to A F, and D G parallel to A E. —— Then, by the Line moving along the Line AE, will the curvilineal Space A E I and Rectangle D E be generated; and, by the Line moving along the Line A F, the curvilineal Space A F I and Rectangle C F will be generated. Now, *before* the Line moving along the Line A E arrives at the Situation C H, it is evident, that, the curvilineal Space A E I will increase *slower*, or flow with a *less* Degree of Velocity than the Rectangle D E, and *afterwards* with a *greater*; therefore, *at* the Term C B, they will flow with *one* and the *same* Degree of Velocity. So likewise, the curvilineal Space A F I will flow or increase *slower*, or with a *less* Degree of Velocity, than the Rectangle C F, *before* the generating Line, moving along

along the Line A F, comes to the Situation D G; and *afterwards*, *faster*, or with a *greater* Degree of Velocity; and therefore, *at* the Term D B, they will increase or flow with an *equal* Degree of Velocity. That is, the *Fluxion* of the curvilineal Space A E I, is, at the Term C B, equal to the *Fluxion* of the Rectangle D E at the said Term C B; and, the *Fluxion* of the curvilineal Space AFI, is, at the Term D B, equal to the *Fluxion* of the Rectangle C F at the same Term D B. But, by *Article* 10. the *Fluxion* of the Rectangle D E, at the Term C B, is equal to C B drawn into the *Fluxion* of A C; and the *Fluxion* of the Rectangle C F, at the Term D B, is equal to D B drawn into the *Fluxion* of A D. Hence, therefore, the *Fluxion* of the flowing Rectangle A C B D (which is the Sum of the two curvilineal Spaces A C B and A D B,) is equal to the Sum of C B drawn into the *Fluxion* of A C, added to D B drawn into the *Fluxion* of A D: that is, if we put A C or D B $= x$, and A D or C B $= y$, the *Fluxion* of the Rectangle xy will be $= \dot{x}y + x\dot{y}$.

2°. HENCE, and from *Article* 8. the *Fluxion* of axy is $= a\dot{x}y + ax\dot{y}$; for, since xy is always in the fixt Ratio to axy, of 1 to

to a; therefore the *Fluxion* of xy is to the *Fluxion* of axy, as 1 to a: that is, $1 : a :: \dot{x}y + x\dot{y} : a\dot{x}y + ax\dot{y} =$ the *Fluxion* of axy.

3°. AND, that the *Fluxion* of xyz is $= \dot{x}yz + x\dot{y}z + xy\dot{z}$, may thus be proved. Let the Length, Breadth, and Depth, of a Parallelopipedon, be repreſented by x, y, and z, reſpectively. Now, this Parallelopipedon, will be equal to three Pyramids, whoſe Baſes are xy, xz, and yz, and Altitudes z, y, and x, reſpectively: And therefore, the *Sum* of the *Fluxions* of theſe Pyramids, will be equal to the *Fluxion* of the Parallelopipedon xyz. Let either of the Pyramids be repreſented as in the *Fig.* by A E H or



ACB, which ſuppoſe to be generated by the variable Plane A F moving with a paral-

lel Motion along the indefinite right Line AE; and, at the ſame Time, let the Parallelopipedon ADGE (whoſe Face AD or EG is equal and ſimilar to the Baſe of the Pyramid at the Term CB,) be generated by the Plane AD, which is coincident with the Plane AF. Now, it is plain, that, the Pyramid will increaſe *ſlower*, or flow with a *leſs* Degree of Velocity, than the Parallelopipedon, *before* the generating Planes arrive at the Term CB; and *afterwards*, *faſter*, or with a *greater* Degree of Velocity: Therefore, *at* the ſaid Term CB, they will flow, or be generated, with the *ſame* or an *equal* Degree of Velocity; But, it is likewiſe plain, that, the Velocity with which the Parallelopipedon is generated, is equal to its Face AD or CB drawn into the Velocity of its Motion along the Line or Side AE; therefore, the Velocity with which the Pyramid is generated, is, at the Term CB, equal to its Baſe CB drawn into the Velocity of its Motion, at the Point C, along the Side AE: that is, the *Fluxion* of the Pyramid ACB, is equal to its Baſe CB drawn into the *Fluxion* of its Altitude AC. Hence then, when the Baſe is xy and Altitude z, the *Fluxion* of the Pyramid is $=xy\times\dot{z}=xy\dot{z}$; and, when the

the Base is xz and Altitude y, the *Fluxion* of it is $=xz\dot{x}\dot{y}=x\dot{y}z$; and, lastly, when the Base is yz and Altitude x, its *Fluxion* is $=yz\times\dot{x}=\dot{x}yz$. Consequently, $\dot{x}yz+x\dot{y}z+xy\dot{z}$ (which is the *Sum* of the *Fluxions* of the three Pyramids,) is $=$ the *Fluxion* of the Parallelopipedon xyz.

Or thus. PUT $xy=v$; then $xyz=vz$, and (*Art.* 8. and *Dem.* 1°.) $\dot{v}=\dot{x}y+x\dot{y}$; and, the *Fluxion* of vz, *viz.* $\dot{v}z+v\dot{z}$ is $=$ the *Fluxion* of xyz: Therefore, by Restitution, or writing xy for v, and $\dot{x}y+x\dot{y}$ for $\dot{v}$, we shall have the *Fluxion* of $xyz=\dot{x}yx+x\dot{y}z+xy\dot{z}$; as before.

4°. AND that the *Fluxion* of $axyz$ is $=a\dot{x}yz+ax\dot{y}z+axy\dot{z}$, follows from *Art.* 8. and the above *Dem.* 3°. for since $xyz:axyz::1:a$, therefore $1:a::$ *Fluxion* of xyz : *Fluxion* of $axyz$; i. e. $1:a::\dot{x}yz+x\dot{y}z+xy\dot{z}:a\dot{x}yz+ax\dot{y}z+axy\dot{z}=$ the *Fluxion* of $axyz$.

HENCE,

12. THE *Fluxion* of xx is $=\dot{x}x+x\dot{x}$; the *Fluxion* of axx is $=a\dot{x}x+ax\dot{x}$; the *Fluxion* of xxx is $=\dot{x}xx+x\dot{x}x+xx\dot{x}$; the *Fluxion* of $axxx$ is $=a\dot{x}xx+ax\dot{x}x+axx\dot{x}$: that is, the *Fluxion* of x^2 is $=2x\dot{x}$; the *Fluxion* of ax^2

is $= 2ax\dot{x}$; the *Fluxion* of x^3 is $= 3x^2\dot{x}$ the *Fluxion* of ax^3 is $= 3ax^2\dot{x}$. And, if m represents any *affirmative* or *positive* Whole-number, the *Fluxion* of x^m will be $= mx^{m-1}\dot{x}$; and the *Fluxion* of ax^m will be $= max^{m-1}\dot{x}$.

RULE III.

13. *To find the* Fluxion *of a* Fraction.

MULTIPLY the Denominator into the *Fluxion* of the Numerator; from the Product of which, subtract the Numerator drawn into the *Fluxion* of the Denominator: Then, divide the Remainder by the Square of the Denominator; and you will have the *Fluxion* of the *Fraction* required.

Thus, the *Fluxion* of $\frac{x}{y}$ is $= \frac{\dot{x}y - x\dot{y}}{y^2}$.

For, putting $z = \frac{x}{y}$; then $yz = x$; and the *Fluxion* of this Equation (*Art.* 8. and 11.) is $\dot{y}z + y\dot{z} = \dot{x}$; therefore, by Transposition $y\dot{z} = \dot{x} - \dot{y}z$, and, by Division, $\dot{z} = \frac{\dot{x} - \dot{y}z}{y}$; that is, (by Restitution, or writing $\frac{x}{y}$ for z its equal,)

qual,) $\dot{z}=\frac{\dot{x}-\dot{y}\times\frac{x}{y}}{y}=\frac{\dot{x}y-x\dot{y}}{y^2}=$ the *Fluxion* of $\frac{x}{y}$.

Alfo, the *Fluxion* of $\frac{1}{x}$ is $=\frac{-\dot{x}}{x^2}$. For, if we put $z=\frac{1}{x}$; then $xz=1$, and the *Fluxion* of xz will be $=$ the *Fluxion* of 1, which (*Art.* 7.) is $=0$; i. e. (*Art.* 11.) $\dot{x}z+x\dot{z}=0$: Therefore, $x\dot{z}=-\dot{x}z$, and $\dot{z}=\frac{-\dot{x}z}{x}$ (which, by writing for z its equal $\frac{1}{x}$, is) $=\frac{-\dot{x}}{x^2}=$ the *Fluxion* of $\frac{1}{x}$.

And, after the fame Manner, it may be proved, that, the *Fluxion* of $\frac{1}{x^2}$ is $=\frac{-2x\dot{x}}{x^4}$, i. e. $=\frac{-2\dot{x}}{x^3}$. Thus, put $z=\frac{1}{x^2}$; then, $zx^2=1$, and the *Fluxion* of this, (by *Art.* 7. 11. and 12.) is $\dot{z}x^2+2zx\dot{x}=0$; therefore, $\dot{z}x^2=-2zx\dot{x}$, and $\dot{z}=\frac{-2zx\dot{x}}{x^2}$ (which, by writing $\frac{1}{x^2}$ for its Value z, is) $=\frac{-2x\dot{x}}{x^4}=\frac{-2\dot{x}}{x^3}$ $=$ the *Fluxion* of $\frac{1}{x^2}$.

And

And thus it may be proved, that, the *Fluxion* of $\frac{1}{x^3}$ is $= \frac{-3x^2\dot{x}}{x^6} = \frac{-3\dot{x}}{x^4}$; that the *Fluxion* of $\frac{1}{x^4}$ is $= \frac{-4x^3\dot{x}}{x^8} = \frac{-4\dot{x}}{x^5}$: And so on.

But, by the *New Notation* (*Article* 9.) $\frac{1}{x}$ is $= x^{-1}$; $\frac{1}{x^2}$ is $= x^{-2}$; $\frac{1}{x^3}$ is $= x^{-3}$; $\frac{1}{x^4}$ is $= x^{-4}$; &c. And, $\frac{-\dot{x}}{x^2}$ is $= -x^{-2}\dot{x}$; $\frac{-2\dot{x}}{x^3}$ is $= -2x^{-3}\dot{x}$; $\frac{-3\dot{x}}{x^4}$ is $= -3x^{-4}\dot{x}$; $\frac{-4\dot{x}}{x^5}$ is $= -4x^{-5}\dot{x}$; &c.

Hence,

14. The *Fluxion* of x^{-1} is $= -x^{-2}\dot{x}$; the *Fluxion* of x^{-2} is $= -2x^{-3}\dot{x}$; the *Fluxion* of x^{-3} is $= -3x^{-4}\dot{x}$; the *Fluxion* of x^{-4} is $= -4x^{-5}\dot{x}$; &c. And, if m be made to represent any *Negative* Whole-number, the *Fluxion* of x^m will be $= mx^{m-1}\dot{x}$.

Rule IV.

15. *To find the* Fluxion *of an Expression compounded of different Terms or Quantities connected together by the Signs* + *and* —.

Find

FIND the *Fluxion* of each Term by the preceding *Rules*; which connect together with the Signs of the respective Terms; and you will have the *Fluxion* required.

Thus the *Fluxion* of $ax+y^2-z$ is $=a\dot{x}+2y\dot{y}-\dot{z}$. For, putting $ax+y^2-z=v$; then $ax+y^2=v+z$: And, it is evident, the *Sum* of the *Fluxions* of ax and y^2 must be equal to the *Sum* of the *Fluxions* of v and z; that is, $a\dot{x}+2y\dot{y}=\dot{v}+\dot{z}$; therefore, by Transposition, $a\dot{x}+2y\dot{y}-\dot{z}=\dot{v}=$ the *Fluxion* of $ax+y^2-z$.

And, the *Fluxion* of $x-a^2y+xy$ is $=\dot{x}-a^2\dot{y}+\dot{x}y+x\dot{y}$. And, the *Fluxion* of $ax^2+x-\frac{x}{y}$ is $=2ax\dot{x}+\dot{x}-:\frac{\dot{x}y-x\dot{y}}{y^2}$.

RULE V.

16. *To find the* Fluxion *of any* Power *of a* Fluent *or* flowing Quantity; *whether the* Index *be Integral or Fractional, Affirmative or Negative.*

MULTIPLY the whole Term by the *Index* of the Power; then subtract 1 from the said *Index*, and multiply this whole Term by the *Fluxion* of the Root of the Fluent: and

and you will have the *Fluxion* of the *Power* required.

Thus, the *Fluxion* of $\overline{x+x^2}|^3$ is $= 3\times\overline{x+x^2}|^{3-1}\times\overline{\dot{x}+2x\dot{x}}$, i. e. $=\overline{x+x^2}|^2\times\overline{3\dot{x}+6x\dot{x}}$; the *Fluxion* of $\overline{a+x}|^{-2}$ is $=-2\times\overline{a+x}|^{-2-1}\times\dot{x}$, i. e. $=\overline{a+x}|^{-3}\times-2\dot{x}$; the *Fluxion* of $\overline{2ax-x^2}|^{\frac{1}{2}}$ is $=\frac{1}{2}\times\overline{2ax-x^2}|^{\frac{1}{2}-1}\times\overline{2a\dot{x}-2x\dot{x}}$, i. e. $=\overline{2ax-x^2}|^{-\frac{1}{2}}\times\overline{a\dot{x}-x\dot{x}}$, i. e. (becaufe by *Article* 9. $\overline{2ax-x^2}|^{-\frac{1}{2}}$ is $=\frac{1}{\overline{2ax-x^2}|^{\frac{1}{2}}}$,) $=\frac{a\dot{x}-x\dot{x}}{\overline{2ax-x^2}|^{\frac{1}{2}}}$; the *Fluxion* of $\overline{x-a}|^{-\frac{2}{3}}$ is $=\frac{2}{3}\times\overline{x-a}|^{-\frac{2}{3}-1}\times\dot{x}$, i. e. $=-\frac{2}{3}\times\overline{x-a}|^{-\frac{5}{3}}\times\dot{x}$, i. e. $=\frac{1}{\overline{x-a}|^{\frac{5}{3}}}\times-\frac{2}{3}\dot{x}$, i. e. $=\frac{-2\dot{x}}{3\cdot\overline{x-a}|^{\frac{5}{3}}}$. And, Univerfally, the *Fluxion* of $x^{\frac{m}{n}}$ is $=\frac{m}{n}\times x^{\frac{m}{n}-1}\times\dot{x}$, i. e. $=\frac{m}{n}x^{\frac{m}{n}-1}\dot{x}$; where, either m or n, or both, may be either Integral or Fractional, Affirmative or Negative: So that, the Index $\frac{m}{n}$, expreffes, or reprefents, any affirmative or negative Whole-number or Fraction whatfoever.

For,

For, by *Art.* 12. and 14. if m repreſents any affirmative or negative Whole-number, the *Fluxion* of x^m will be $= mx^{m-1}\dot{x}$; therefore, if we ſuppoſe $y = x^{\frac{m}{n}}$, or, which is the ſame, $y^n = x^m$ (n being any affirmative or negative Whole-number likewiſe,) by *Art.* 8. we have $ny^{n-1}\dot{y} = mx^{m-1}\dot{x}$, which divided by ny^{n-1}, makes $\dot{y} = \frac{mx^{m-1}\dot{x}}{ny^{n-1}}$, i. e. (by writing $x^{\frac{m}{n}}$ for y its Value,) $\dot{y} = \frac{mx^{m-1}\dot{x}}{n \times \overline{x^{\frac{m}{n}}}|^{n-1}} = \frac{m}{n} \times$

$$\frac{x^{m-1}\dot{x}}{x^{\frac{mn-m}{n}}} = \frac{m}{n} \times x^{m-1-:\frac{mn-m}{n}}\dot{x} = \frac{m}{n}x^{\frac{m}{n}-1}\dot{x},$$

which is $=$ the *Fluxion* of $x^{\frac{m}{n}}$, as before; therefore, &c. And that m or n, or both, may repreſent any Fractions as well as Whole-numbers, is plain, ſince this $\frac{m}{n}$ expreſſes the Quotient of any Whole-number divided by another, and may be taken for a *New* m or n; and ſo on *ad infinitum.*

RULE VI.

17. *To find the* Fluxion *of a* Logarithm.

THE *Fluxion* of the Hyperbolic Logarithm of any Quantity, is equal to the *Fluxion* of that Quantity, divided by the Quantity itſelf.

(This we shall hereafter demonstrate. *See* Appendix, *Quest.* 11.)

Thus, the *Fluxion* of the Hyp. Log. of x is $= \frac{\dot{x}}{x}$. The *Fluxion* of the Hyp. Log. of $\overline{a+x}$ is $= \frac{\dot{x}}{a+x}$. The *Fluxion* of the Hyperbolic Logarithm of $\overline{x + \overline{a^2+x^2}|^{\frac{1}{2}}}$ is $= \frac{\dot{x}}{\overline{a^2+x^2}|^{\frac{1}{2}}}$; (for the *Fluxion* of $x+\overline{a^2+x^2}|^{\frac{1}{2}}$ is $= \dot{x} + \frac{x\dot{x}}{\overline{a^2+x^2}|^{\frac{1}{2}}} = \frac{\dot{x}\times\overline{a^2+x^2}|^{\frac{1}{2}}+x\dot{x}}{\overline{a^2+x^2}|^{\frac{1}{2}}} = \frac{\dot{x}}{\overline{a^2+x^2}|^{\frac{1}{2}}} \times \overline{x+\overline{a^2+x^2}|^{\frac{1}{2}}}$, which divided by $x+\overline{a^2+x^2}|^{\frac{1}{2}}$, gives $\frac{\dot{x}}{\overline{a^2+x^2}|^{\frac{1}{2}}}$). The *Fluxion* of the Hyp. Log. of $\frac{a+x}{a-x}$ is $=\frac{2a\dot{x}}{a^2-x^2}$; (for the *Fluxion* of $\frac{a+x}{a-x}$ (*Art.* 13) is $= \frac{\dot{x}\times\overline{a-x}+\dot{x}\times\overline{a+x}}{\overline{a-x}|^2}$, i.e. $=\frac{2a\dot{x}}{\overline{a-x}|^2}$, which divided by $\frac{a+x}{a-x}$, gives $\frac{2a\dot{x}\times\overline{a-x}}{\overline{a-x}|^2\times\overline{a+x}}$, i. e. $\frac{2a\dot{x}}{\overline{a-x}\times\overline{a+x}}$ i. e. $\frac{2a\dot{x}}{a^2-x^2}$). And, the *Fluxion* of the Hyp. Log. of $\overline{a+x+\overline{2ax+x^2}|^{\frac{1}{2}}}$ is $=\frac{\dot{x}}{\overline{2ax+x^2}|^{\frac{1}{2}}}$; (for

(for the *Fluxion* of $a+x+\overline{2ax+x^2}\big|^{\frac{1}{2}}$ is $=\dot{x}+$

$$\frac{a\dot{x}+x\dot{x}}{\overline{2ax+x^2}\big|^{\frac{1}{2}}}=\frac{\dot{x}\times\overline{2ax+x^2}\big|^{\frac{1}{2}}+a\dot{x}+x\dot{x}}{\overline{2ax+x^2}\big|^{\frac{1}{2}}}=$$

$\dfrac{\dot{x}}{\overline{2ax+x^2}\big|^{\frac{1}{2}}}\times\overline{a+x+\overline{2ax+x^2}\big|^{\frac{1}{2}}}$, which divided by $a+x+\overline{2ax+x^2}\big|^{\frac{1}{2}}$, gives $\dfrac{\dot{x}}{\overline{2ax+x^2}\big|^{\frac{1}{2}}}$).

NOW, as the Hyp. Log. of 10 (*viz.* 2.302585092 9, &c. See *Appendix*, Question [illegible].) : is to the common Log. of 10 (*viz.* 1.) : : ſo is the Hyp. Log. of any Number, x, : to the common Log. of that ſame Number, x; that is, if we put L=2.3025 &c. As L : 1 : : Hyp. Log. of x : common Log. of x (and therefore *Article* 8.) : : *Fluxion* of Hyp. Log. of x : *Fluxion* of common Log. of x.

12

HENCE we have $L : 1 :: \dfrac{\dot{x}}{x} : \dfrac{\dot{x}}{Lx}=$ the *Fluxion* of the common Log. of x; or, becauſe $\dfrac{1}{L}=0.4342944819$ &c. if we put this $\dfrac{1}{L}=M$, then the *Fluxion* of the common Log. of x (*viz.* $\dfrac{\dot{x}}{Lx}$) will be $=\dfrac{\dot{x}}{x}\times M$; i.e. the *Fluxion* of the Hyp. Log. of any

 Num-

Number, multiplied by (M or) 0.434294 &c. gives the *Fluxion* of the common Log. of that same Number.

18. *Note.* It is sometimes expedient, in Order to find the *Fluxion* of an Equation, to divide it by one, or more, of the variable Quantities contained in most of the Terms: And to substitute single Letters for compound Quantities.

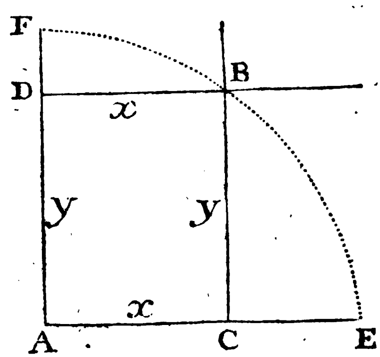
Thus, if the Equation were $xy + a - axy + y^2 = yz$; divide both Sides of it by y: then it will be $x + \frac{a}{y} - ax + y = z$; and the *Fluxion* of this Equation is $\dot{x} - \frac{a\dot{y}}{y^2} - a\dot{x} + \dot{y} = \dot{z}$. —— And, if the Equation were $x - by + \frac{x^2y}{z} = az$; substitute v for $\frac{x^2y}{z}$: then $x - by + v = az$; and this Equation put into *Fluxions*, is $\dot{x} - b\dot{y} + \dot{v} = a\dot{z}$.

19. *Note.* Hitherto we have supposed, when one variable Quantity in any Fluent increases, that the others, if any, increase likewise: But, it often happens, that some of them *decrease*, while the others *increase*; in which Case, the *Fluxions* of the *decreasing* are *Negative* with respect to those of the *increasing* Quantities; and therefore, the Signs of

of the Terms which are affected with them, ought to be *Negative.*

Thus, if whilst x increases, y decreases; the *Fluxion* of the Rectangle $x y$, will not be expressed by $\dot{x}y+x\dot{y}$, but by $\dot{x}y-x\dot{y}$.

For, let the flowing or variable Rectangle A B, be continually increasing by the parallel Motion of the variable Line C B moving from the Situation A F along the Line A E; and, continually diminishing by the Motion of the variable Line D B, parallel to the Line A E, and approaching towards it along



the Line FA; that is, let the Side AC=DB continually increase, while the Side AD= C B continually decreases. Now (putting AC=x, and AD=y,) it follows from *Art.* 11. *Dem.* 1°. that the *Fluxion* of the *Increase* will be=$\dot{x}y$, and that the *Fluxion* of the *Decrease*

crease will be$=x\dot{y}$; therefore the *Fluxion* of the *Decrease*, subtracted from the *Fluxion* of the *Increase*, leaves $\dot{x}y-x\dot{y}=$the *Fluxion* of the Rectangle xy; or, the *Velocity* with which it flows.

UNDERSTAND the same of the *Fluxion* of any *Fluent* of a like Nature whatsoever.

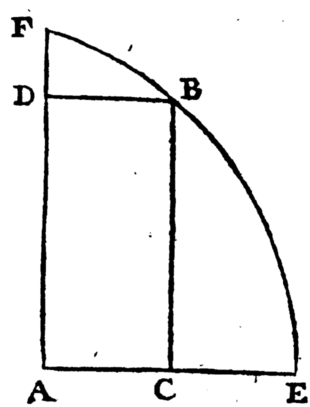
IF what has been already delivered, be thoroughly understood, (and it is indispensibly necessary it should;) it is hoped, the Learner will meet with but few Difficulties in the Application thereof: to which we now proceed.

CHAP. III.

Of finding the Maxima *and* Minima *(or* Greatest *and* Least) *of variable Quantities.*

20. A MAXIMUM, is the *greatest* Magnitude of a variable Quantity; which naturally increases before it arrives at that Magnitude, and as naturally decreases afterwards: And, a MINIMUM, is the *least* Magnitude of a variable Quantity; which naturally decreases before it arrives at that Magnitude, and afterwards as naturally increases.

creases. Therefore, at that certain Term, where a Quantity becomes a MAXIMUM or MINIMUM, it can neither be increasing nor decreasing, and consequently its *Fluxion* is $= 0$.



21. Thus (for an Illustration,) in the curvilineal Space A E F, let the equal and parallel Sides A C and D B, of the inscribed and flowing Rectangle A B, be continually *increasing*, while the other equal and parallel Sides AD and C B are continually *decreasing*; that is, let the Rectangle be *increased* by the Motion of the variable and decreasing Side C B, and *decreased* by the Motion of the variable and increasing Side D B. Now, it is evident, that, while the Rectangle is increasing faster or with a greater Degree of Velocity, by the Motion of the Line CB, than it is decreasing by the Motion of the Line DB, it will be continually *increasing*; and, that, when it is decreasing faster, or with a greater Degree of Velocity, by the Motion of the Line DB, than

than it is increasing by the Motion of the Line CB, it will be continually *decreasing*: Therefore, when these two Degrees of Velocity become equal, the Rectangle will be a *Maximum*; which will then be neither *increasing* nor *decreasing*; or, the *Velocity*, or *Fluxion*, with which it is increasing (which is Affirmative,) added to the *Velocity*, or *Fluxion*, with which it is decreasing (which is Negative,) gives the *Velocity*, or *Fluxion*, with which it flows $=$ o.——And, it being manifest, that, when the Rectangle AB becomes a *Maximum*, the Sum of the Spaces FDB and BCE will be a *Minimum*; therefore, when the Sum of the said Spaces is a *Minimum*, the *Fluxion* of the increasing Space FDB, will be equal to the *Fluxion* of the decreasing Space ECB; and therefore, the former being Affirmative, and the latter Negative, their Sum is $=$ o.

22. Now, when a Quantity becomes a *Maximum* or *Minimum*, and is expressed by two or more affirmative and negative Terms, in which is contained but one variable Letter; it is evident the *Fluxions* of the affirmative and negative Terms will be equal, since their Sum $=$ o. —— And, when in the Expression for the *Fluxion* of a *Maximum* or *Mini-*

Minimum, there are two or more *fluxionary* Letters, each contained in both affirmative and negative Terms; the Sum of the Terms affected with either of them will likewise be$=$o. Thus, if a *Minimum* be expressed by $ax - 2xy + by$, whose *Fluxion* is $a\dot{x} - 2\dot{x}y - 2x\dot{y} + b\dot{y} = 0$; then it will be $a\dot{x} - 2\dot{x}y = 0$, and $b\dot{y} - 2x\dot{y} = 0$: For, in Order for any Expression to be a *Maximum* or *Minimum*, in which are contained two or more variable Quantities, it must produce either a *Maximum* or *Minimum*, if but one of those Quantities is supposed to be variable: And therefore, if in this Expression of the *Minimum*, y be made invariable, the *Fluxion* of it will be $a\dot{x} - 2\dot{x}y = 0$; and if x be invariable, it will be $-2x\dot{y} + b\dot{y} = 0$.

23. *Note* 1. IN curvilineal Spaces, it is, in effect, the same Thing to seek the greatest Area that can be contained under a given Perimeter, as to seek a given Area under the least Perimeter: And, the same holds good, in respect of Solids and their Surfaces.

Note 2. WHEN any Quantity is a *Maximum* or *Minimum*, all the Powers of it will be so too; as will also, the Product or Quotient arising from its being multiplied or divided by any invariable or given Quantity.

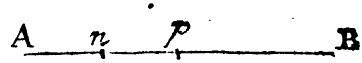
 Thus,

Thus, if $\frac{a}{b}\times\overline{ax-x^2}|^{\frac{1}{3}}$, or $\overline{\overline{a^2+x^2}|^{\frac{1}{2}}-\frac{b}{c}x}|^2\times d$, represents a *Maximum* or *Minimum*, then $ax-x^2$, or $\overline{a^2+x^2}|^{\frac{1}{2}}-\frac{b}{c}x$, will also be a *Maximum* or *Minimum*; and consequently, its *Fluxion* must be $=0$.

Note 3. GENERALLY, when a variable Quantity admits of a *Maximum*, its *Minimum* is nothing: and, when it admits of a *Minimum*, its *Maximum* is infinite.

EXAMPLE I.

24. *To find the Point* p, *in the given right Line* A B, *where the Rectangle of the Parts* Ap *into* pB *is a* Maximum, *or greater than any other Rectangle* A n *into* n B.

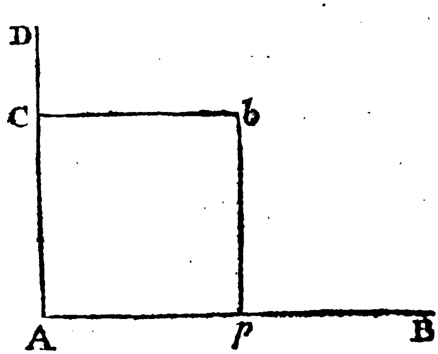


PUT the given right Line A B $=a$, and A$p=x$; then will p B be $=a-x$; and A$p\times p$B$=x\times\overline{a-x}=ax-x^2=$ a *Maximum*, by the *Quest.* therefore, (the *Fluxion* of a *Maximum* being $=0$,) the *Fluxion* of $ax-x^2$ is $=0$; that is (*Art.* 15.) $a\dot{x}-2x\dot{x}=0$. Now, this *Fluxion* divided by $\dot{x}$, makes $a-2x=0$; therefore, $a=2x$, and $x=\frac{1}{2}a$. Consequently, the Rectangle of the Parts Ap

Ap and pB is a *Maximum*, or the greateſt poſſible, when theſe Parts are equal.

Or thus. PUT $Ap=x$, and $pB=y$; then $Ap\times pB=xy=$ a *Maximum*. Now, it is evident, that, if x increaſes, y muſt decreaſe; therefore (*Art.* 19.) $\dot{x}$ and $\dot{y}$ are Negative to each other, and conſequently, when xy is a *Maximum*, the *Fluxion* of it will be $\dot{x}y-x\dot{y}=0$. But, it is likewiſe evident, that, the Velocity with which x increaſes, is equal to the Velocity with which y decreaſes; or that, the *Increment* of x is equal to the *Decrement* of y; that is, $\dot{x}=\dot{y}$: Therefore, by ſtriking both $\dot{x}$ and $\dot{y}$ out, in the above *Fluxion* of xy, we ſhall have $y-x=0$; and therefore $x=y$; as before.

Or thus. IN the variable Rectangle Ab,

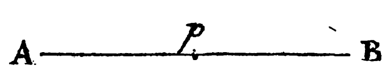


let the Side pb (which is equal and parallel to the Side AC) always be equal to pB: then

then while the Rectangle is increasing by the Motion of the decreasing Side pb, moving from A towards B, it will be decreasing by the Motion of the increasing Side bC, moving from D towards A. Now, since the *Velocities* of the Points p and C, or the *Fluxions* of the Sides Ap and AC, are always equal, (as they evidently must be, because the Sum of these Sides is always the same;) therefore, as long as the Side Ap continues *less* than the Side AC (or pb) the Rectangle will be continually *increasing*; and after the Side Ap becomes *equal* to the Side pb, it will be continually *decreasing*: Therefore, when the Rectangle is a *Maximum*, or when it is neither increasing nor decreasing, the Sides Ap and pb are equal to each other, or $Ap = pb = pB$; as before.

EXAMPLE II.

25. *To find the Point* p *in the given right Line* AB, *when* $\overline{Ap}|^m \times \overline{pB}|^n$ *is a* Maximum.



PUT the given right Line $AB = a$, and $Ap = x$; then $pB = a - x$, and $\overline{Ap}|^m \times \overline{pB}|^n = x^m \times \overline{a - x}|^n =$ a *Maximum*; therefore the *Fluxion*

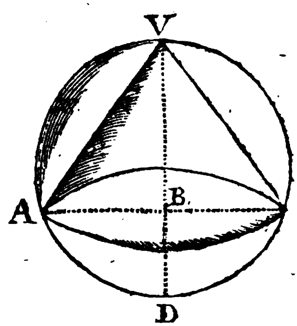
Fluxion of $x^m \times \overline{a-x}|^n$ is $=0$; that is (*Art.* 11. and 16.) $mx^{m-1}\dot{x} \times \overline{a-x}|^n + n.\overline{a-x}|^{n-1} \times -\dot{x} \times x^m = 0$ (i. e.) $mx^{m-1}\dot{x} \times \overline{a-x}|^n - nx$ $\overline{a-x}|^{n-1}\dot{x} \times x^m = 0$: Now, by dividing this Equation by $x^{m-1}\dot{x}$, we have $m.\overline{a-x}|^n - nx$ $\overline{a-x}|^{n-1}x = 0$; therefore $m.\overline{a-x}|^n = nx$ $\overline{a-x}|^{n-1}x$; which Equation divided by $\overline{a-x}|^{n-1}$, gives $m.\overline{a-x} = nx$, i. e. $ma - mx = nx$; therefore $mx + nx = ma$, and $x = \frac{m}{m+n}a$; whence the Point p is determined.

Or thus. PUT $Ap = x$, and $pB = y$; then $y^n \times x^m =$ a *Maximum*; which put into *Fluxions*, (because when y increases x decreases, or the *Fluxions* of y^n and x^m are negative to each other; *Art.* 19.) is $ny^{n-1}\dot{y} \times x^m - mx^{m-1}\dot{x} \times y^n = 0$: But $\dot{x}$ and $\dot{y}$ are equal, (*Art.* 24.) therefore, throw both $\dot{x}$ and $\dot{y}$ out; then it will be $ny^{n-1}x^m - mx^{m-1}y^n = 0$, or $ny^{n-1}x^m = mx^{m-1}y^n$: Now, by dividing both Sides of this Equation by $y^{n-1}x^{m-1}$, we have $nx = my$; therefore, $m : n :: x : y$. So that, the Segments Ap and pB, are in direct Proportion to the Powers in the *Maximum*.

EXAM-

EXAMPLE III.

26. *To find a Cone inscribed in a given Sphere, whose convex Surface shall be a* Maximum.



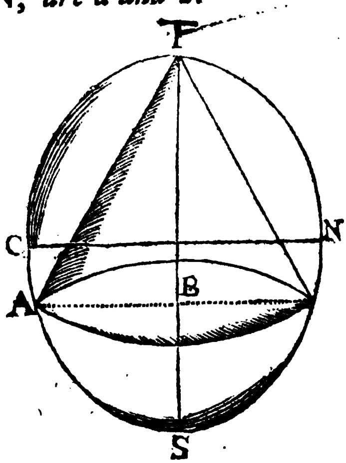
PUT the Diameter of the given Sphere (VD) $=a$; the Altitude of the inscribed Cone (VB) $=x$; and 3.14159 &c. $=c$: then B D $=a-x$. Now, by a well known Property of Circles, $VB\times BD=\overline{BA}|^2$; that is, $x\times\overline{a-x}=ax-x^2=\overline{BA}|^2$: And, by 47 E. 1. $AV=\overline{\overline{VB}|^2+\overline{BA}|^2}|^{\frac{1}{2}}$; that is, $AV=\overline{x^2+ax-x^2}|^{\frac{1}{2}}=\overline{ax}|^{\frac{1}{2}}$: But, $BA\times c=\overline{ax-x^2}|^{\frac{1}{2}}\times c=\overline{c^2ax-c^2x^2}|^{\frac{1}{2}}=\frac{1}{2}$ the Circumference of the Cone's Base; which drawn into its slant Height ($AV=$) $\overline{ax}|^{\frac{1}{2}}$, is $\overline{c^2ax-c^2x^2}|^{\frac{1}{2}}\times\overline{ax}|^{\frac{1}{2}}=\overline{c^2a^2x^2-c^2ax^3}|^{\frac{1}{2}}=$ its curve Superficies, which by the *Question* must be a *Maximum*; therefore, $c^2a^2x^2-c^2ax^3=$ the *Square* of a *Maximum*, or, is a *Maximum* also, (*Art.*23. *Note* 2.) and its *Fluxion* therefore is $=0$; i. e.

i. e. $2c^2a^2x\dot{x}-3c^2ax^2\dot{x}=0$; which divided by $c^2ax\dot{x}$, gives $2a-3x=0$; therefore $3x=2a$, and $x=\frac{2}{3}a$. So that, the convex Surface of the Cone will be the greatest, when its Altitude is $=\frac{2}{3}$ of the Sphere's Diameter.

27. *N. B.* There was no need of introducing c in the above; for $VA \times AB =$ a *Maximum*, the Radii of Circles being as their Circumferences: And therefore, in such like Cases, we shall omit inserting it, as often as may be, in the following Examples.

EXAMPLE IV.

28. *To find the largest Cone that can be inscribed in a given Spheroid,* viz. *whose Transverse and Conjugate Diameters,* T S *and* CN, *are* a *and* b.



PUT

PUT the Altitude of the inscribed Cone, BT, $=x$; then B S $=$ T S $-$ T B $= a - x$. Now, by a well known Property of Ellipses, $\overline{TS}|^2 : \overline{CN}|^2 :: TB \times BS : \overline{AB}|^2$, i. e. $a^2 : b^2 :: x \times \overline{a-x} : \frac{ab^2x - b^2x^2}{a^2} = \overline{BA}|^2$, which (*Art.* 27.) drawn into $(\frac{1}{3}BT)\frac{1}{3}x$, gives $\frac{ab^2x^2 - b^2x^3}{3a^2} =$ a *Maximum:* Consequently (*Art.* 23. *Note* 2.) $ax^2 - x^3 =$ a *Maximum*, whose *Fluxion* is$=$o, i. e. $2ax\dot{x} - 3x^2\dot{x} = 0$; which divided by $x\dot{x}$, gives $2a - 3x = 0$: Consequently $3x = 2a$, and $x = \frac{2}{3}a$; that is, the Altitude of the Cone must be$=\frac{2}{3}$ of the Diameter of the Spheroid.

EXAMPLE V.

29. *To find the internal Dimensions of a cylindrical Cup, whose Capacity is given* $=$ a, *when the Cup is made with the least possible Quantity of Silver of a given thickness.*

PUT the Diameter$=x$, and .78539 &c. $=c$; then $cx^2 =$the Area of the Bottom, and therefore $\frac{a}{cx^2} =$the Altitude: But $4cx$ being $=$ the Circumference of the Bottom; therefore

fore $4cx\times\frac{a}{cx^2}=\frac{4a}{x}=$the inſide curve Superficies: Hence $cx^2+\frac{4a}{x}=$the whole inſide Superficies; which, becauſe the Quantity of Silver is the leaſt poſſible, is a *Minimum*; therefore the *Fluxion* of $cx^2+\frac{4a}{x}$ muſt be $=$ o, i. e. $2cx\dot{x}-\frac{4a\dot{x}}{x^2}=0$; which multiplied by x^2, is $2cx^3\dot{x}-4a\dot{x}=0$; and this divided by $2\dot{x}$, is $cx^3-2a=0$; therefore $cx^3=2a$, and $x=\overline{\frac{2a}{c}}\Big|^{\frac{1}{3}}=$ the Diameter; which ſubſtituted for x, makes the above $\frac{a}{cx^2}=\frac{a}{c\times\overline{\frac{2a}{c}}\Big|^{\frac{2}{3}}}$

$$=\frac{a\times\overline{\frac{2a}{c}}\Big|^{\frac{1}{3}}}{c\times\frac{2a}{c}}=\tfrac{1}{2}\times\overline{\frac{2a}{c}}\Big|^{\frac{1}{3}}=$$ the Altitude. So that, the Diameter is to the Altitude, as 2 to 1.

Or thus. PUT the Diameter $=x$, Altitude $=y$, and .78539 &c. $=c$; then the whole inſide Superficies will be $=cx^2+4cxy$ $=$ a *Minimum*; the *Fluxion* of which (ſuppoſing x to increaſe, and therefore y to de-

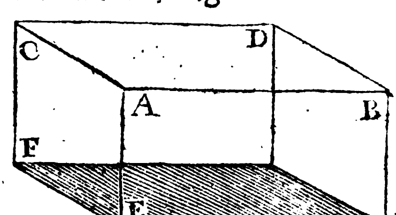
 creaſe,

crease, or $\dot{x}$ and $\dot{y}$ Negative to each other, as they evidently must be,) is $2cx\dot{x}+4c\dot{x}y-4cx\dot{y}=0$: But, by the *Quest.* $cx^2y=a$; which in *Fluxions* (because the *Fluxion* of a is $=0$,) is $2cx\dot{x}y-cx^2\dot{y}=0$; therefore, by Transposition, $cx^2\dot{y}=2cx\dot{x}y$, which Equation divided by cx^2, makes $\dot{y}=\frac{2cx\dot{x}y}{cx^2}=\frac{2\dot{x}y}{x}$: Now, this Value of $\dot{y}$ being substituted for it in the above *Fluxion* of the *Minimum*, makes it $2cx\dot{x}+4c\dot{x}y-8c\dot{x}y=0$; which divided by $2c\dot{x}$, is $x+2y-4y=0$, i. e. $x-2y=0$: Consequently $x=2y$; as before.

EXAMPLE VI.

30. *To find the internal Dimensions of a Cistern in the Form of a rectangular Solid, i.e. whose Bottom and all four Sides are rectangular; when its Capacity is* = a, *and made with the least possible Quantity of Lead of a given thickness.*

PUT the inside Length A B or C D $=x$,



Breadth AC or BD $=y$, and Depth AE or

CF

CF $=z$; then $xyz=a$, and therefore $x=\frac{a}{yz}$. Now, the inside Superficies of the Bottom and four Sides, is $=$ AB×BD$+$2BA×AE $+$2CA×AE, i.e. $=xy+2xz+2yz$ (or, by substituting $\frac{a}{yz}$ for x,) $=\frac{a}{z}+\frac{2a}{y}+2yz$; which, because the Cistern is made with the least possible Quantity of Lead, is $=$ a *Minimum*; and therefore its *Fluxion* is $=0$; that is, $-\frac{a\dot{z}}{z^2}-\frac{2a\dot{y}}{y^2}+2\dot{y}z+2y\dot{z}=0$. Now (*Art.* 22.) the Sum of the Terms affected with $\dot{y}$ is $=0$, and the Sum of the Terms affected with $\dot{z}$ is $=0$; that is, $-\frac{2a\dot{y}}{y^2}+2\dot{y}z=0$, and $-\frac{a\dot{z}}{z^2}+2y\dot{z}=0$; the first of which two Equations multiplied by y^2, is $-2a\dot{y}+2y^2\dot{y}z=0$, and this divided by $2\dot{y}$, is $-a+y^2z=0$, $\because y^2z=a$ (which by the above is) $=xyz$; therefore $y=x$: And the second of the said two Equations multiplied by z^2, is $-a\dot{z}+2yz^2\dot{z}=0$, which divided by $\dot{z}$, is $-a+2yz^2=0$, $\because 2yz^2=a=$ (by the above,) xyz; therefore $2z=x$. Hence $x=y=2z$; that is, the Length and Breadth will be equal, and each equal to twice the Depth.

 Or

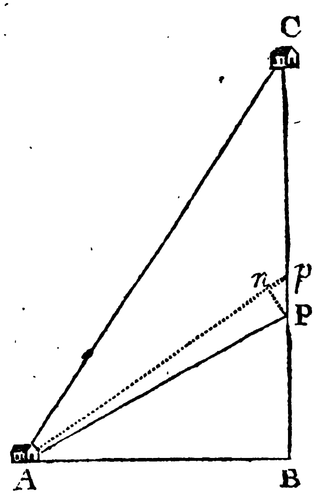
Or thus. THE Length, Breadth, and Depth, being x, y, and z, respectively; and the inside Superficies $= xy + 2xz + 2yz =$ a *Minimum*, as before: If we suppose x and y to increase, then z must decrease; i. e. the *Fluxions* of x and y being Affirmative, the *Fluxion* of z will be Negative; and therefore, the *Fluxion* of the *Minimum* will be $\dot{x}y + x\dot{y} + 2\dot{x}z - 2x\dot{z} + 2\dot{y}z - 2y\dot{z} = 0$: But, by the *Quest.* $xyz = a$, the *Fluxion* of which Equation (the *Fluxion* of a being $= 0$, and the *Fluxion* of z being Negative,) is $\dot{x}yz + x\dot{y}z - xy\dot{z} = 0$; $\therefore \dot{z} = \frac{\dot{x}yz + x\dot{y}z}{xy} = \frac{\dot{x}z}{x} + \frac{\dot{y}z}{y}$, which substituted for $\dot{z}$, in the *Fluxion* of the *Minimum*, makes it $\dot{x}y + x\dot{y} - \frac{2x\dot{y}z}{y} - \frac{2\dot{x}yz}{x} = 0$; therefore (*Art.* 22.) $\dot{x}y - \frac{2\dot{x}yz}{x} = 0$, and $x\dot{y} - \frac{2x\dot{y}z}{y} = 0$: Now, the first of these two Equations multiplied by $\frac{x}{\dot{x}y}$, is $x - 2z = 0$; $\therefore x = 2z$; and the second of them multiplied by $\frac{y}{x\dot{y}}$, is $y - 2z = 0$, $\therefore y = 2z$. So that $x = y = 2z$; as before.

N. B.

N. B. In this last Method of Solution, either of the Quantities x, y, or z, might have been supposed to decrease while the others increase; or, but one of them to increase while the others decrease: and the Conclusions would have come out exactly the same.

EXAMPLE VII.

31. *A Gentleman wants to ride from the City* A *to the City* C, *the Cities being* a *Miles apart: Now from* A *to* B, *which is* b *Miles,*



or from A *to any Place* P *in the Road* C B *which is perpendicular to the Road* B A, *he can ride after the rate of* c *Miles an Hour;* *but*

but from B *to* C *he can ride faster, or after the rate of* d *Miles an Hour. To find* P, *the Place to which he must directly ride, in order to perform his Journey in the least possible Time.*

By 47 E. 1. $BC = \overline{\overline{CA}|^2 - \overline{AB}|^2}|^{\frac{1}{2}} = \overline{a^2 - b^2}|^{\frac{1}{2}}$, which put $= e$; also let $BP = x$; then $PC = e - x$; therefore, by *Question* $\frac{e-x}{d} =$ the Number of Hours the Gentleman will be riding from P to C. Again, by 47 E. 1. $AP = \overline{\overline{PB}|^2 + \overline{BA}|^2}|^{\frac{1}{2}} = \overline{x^2 + b^2}|^{\frac{1}{2}}$; therefore, by *Question* $\frac{\overline{x^2+b^2}|^{\frac{1}{2}}}{c} =$ the Number of Hours he will be riding from A to P. Hence we have $\frac{\overline{x^2+b^2}|^{\frac{1}{2}}}{c} + \frac{e-x}{d} =$ the Number of Hours he will be performing his Journey; which, by *Question* must be a *Minimum*: the *Fluxion* of it therefore is $= 0$; that is, $\frac{\frac{1}{2} \times \overline{x^2+b^2}|^{-\frac{1}{2}} \times 2x\dot{x}}{c} - \frac{\dot{x}}{d} = 0$, i. e. $\frac{\frac{1}{2} \times \frac{1}{\overline{x^2+b^2}|^{\frac{1}{2}}} \times 2x\dot{x}}{c} - \frac{\dot{x}}{d} = 0$, i. e. $\frac{x\dot{x}}{c \times \overline{x^2+b^2}|^{\frac{1}{2}}} - \frac{\dot{x}}{d} = 0$; which multiplied by

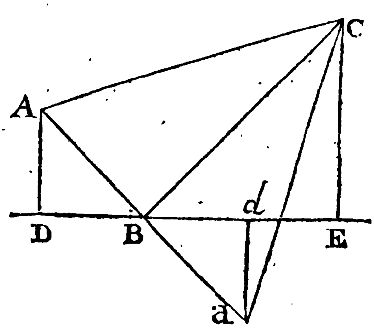
$cd\times\overline{x^2+b^2}|^{\frac{1}{2}}$, gives $d\dot{x}\dot{x}-c\dot{x}\times\overline{x^2+b^2}|^{\frac{1}{2}}=0$; and this transposed and divided by $\dot{x}$, is $dx=c\times\overline{x^2+b^2}|^{\frac{1}{2}}$; the Square of which is $d^2x^2=c^2x^2+c^2b^2$; therefore $d^2x^2-c^2x^2=c^2b^2$, and $x^2=\frac{c^2b^2}{d^2-c^2}$; consequently $x=\frac{cb}{\overline{d^2-c^2}|^{\frac{1}{2}}}=BP$.

Or thus. LET A*p* be supposed indefinitely near to AP, and the little circular Arch P*n* be described with the Radius A P; that is, let P*p* express the nascent or indefinitely small *Increment* of B P, and *np* that of A P. Now, it is evident, if P be the Place to which he must directly ride, that the *Increments* P*p* and *np* must be to each other as *d* to *c*, since then only can they be passed over in the same Time; but, the little Triangle P*np* is (or may be taken as) similar to the right angled Triangle ABP; (for, the Arch P*n* being indefinitely small, may be considered as a little right Line perpendicular to A*p*; so likewise, the Angle PA*p* being indefinitely little, the Angle A*p*B or *np*P may be considered as equal to the Angle A P B, and therefore the Angle *p*P*n* as equal to the Angle P A B:) Consequently, $d:c::$ AP : P B: But, when the Hypothenuse and Perpendicular are *d* and *c*, the Base, by 47 E. 1.

E. 1. will be $\overline{d^2-c^2}|^{\frac{1}{2}}$. Hence, therefore, $\overline{d^2-c^2}|^{\frac{1}{2}}$: (AB) $b :: c : \frac{bc}{\overline{d^2-c^2}|^{\frac{1}{2}}} =$ B P; as before.

Example VIII.

32. *Let the Triangle* ABC *have one Angle* B *in the right Line* DE: *To find a* Maximum *of the Sum of its Perpendiculars* AD *and* CE *dropt from the other two Angles on the right Line aforesaid.*



N. B. AB$=3=a$, BC$=4=b$, and the $\angle$ABC$=90^{\circ}$.

Put DB$=x$; then, by 47 E. 1. AD$=$ $\overline{\overline{AB}|^2-\overline{BD}|^2}|^{\frac{1}{2}}=\overline{a^2-x^2}|^{\frac{1}{2}}$. Now, becauſe the $\angle$ABC is right, the $\angle$CBE is the Complement of the $\angle$ ABD, and therefore is $=$ $\angle$ BAD; conſequently the Triangles ABD

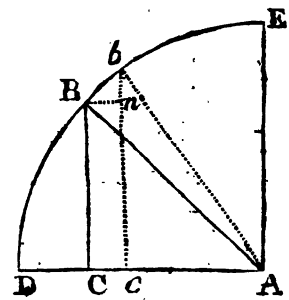
and CBE are similar; therefore AB : BD :: BC : CE, i. e. $a : x :: b : \frac{bx}{a} =$ CE. Hence we have AD + CE $= \overline{a^2 - x^2}|^{\frac{1}{2}} + \frac{bx}{a} =$ a *Maximum*, by *Quest.* the *Fluxion* of which is therefore $= 0$; that is, $\frac{1}{2} \times \overline{a^2 - x^2}|^{-\frac{1}{2}} \times - 2x\dot{x} + \frac{b\dot{x}}{a} = 0$, i. e. $- \frac{x\dot{x}}{\overline{a^2 - x^2}|^{\frac{1}{2}}} + \frac{b\dot{x}}{a} = 0$; which multiplied by $\overline{a^2 - x^2}|^{\frac{1}{2}} \times a$, gives $- ax\dot{x} + \overline{a^2 - x^2}|^{\frac{1}{2}} b\dot{x} = 0$; therefore, by Transposition and dividing by $\dot{x}$, we have $ax = \overline{a^2 - x^2}|^{\frac{1}{2}} b$, by Involution $a^2x^2 = a^2b^2 - b^2x^2$; consequently $a^2x^2 + b^2x^2 = a^2b^2$, and $x^2 = \frac{a^2b^2}{a^2 + b^2}$; therefore $x = \frac{ab}{\overline{a^2 + b^2}|^{\frac{1}{2}}} = 2.4$. Hence AD + CE $(= \overline{a^2 - x^2}|^{\frac{1}{2}} + \frac{bx}{a}) = 1.8 + 3.2 = 5 =$ the Sum of the Perpendiculars, when that Sum is the greatest possible; and which, by 47 E. 1. is = the Hypothenuse AC.

Or (without the help of Fluxions) *thus.* Produce A B to *a*, making aB=BA; draw the Line a C; and perpendicular to D E draw a *d*: Then will the Triangles B a C and B A C be equal and similar; and the Perpendicular a*d* be equal and parallel to the

Perpendicular D A. Now, it is evident, that, the Line a C, in every Situation, will be greater than the Sum of the Perpendiculars a *d* and C E, excepting when it is perpendicular to the Line D E; and that then, the said Perpendiculars will coincide with, and their Sum be equal to it. Therefore, the *Maximum* of the Sum of the Perpendiculars a *d* and C E, or of A D and C E, is equal to the Side of the Triangle, a C or AC; which, (when the ∠aBC or ABC is a right Angle, and a B or B A $=3$, and B C $=4$, as in the present Case,) by 47 E. 1. is $=5$; as before.

Example IX.

33. *To find the greatest right angled Triangle* A C B *that can be inscribed in the given Quadrant* A D E; *the right Angle being formed by the Sine and Cosine* B C *and* C A.



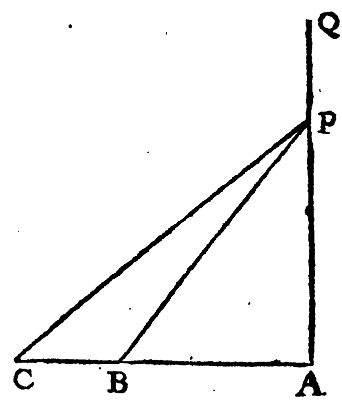
Put the Radius or Hypothenuse A B $=a$, and AC$=x$; then, by 47 E. 1. CB$=\overline{a^2-x^2}|^{\frac{1}{2}}$: therefore, by *Question* $\frac{1}{2}x \times \overline{a^2-x^2}|^{\frac{1}{2}}$ $=$ a

$= \text{a } Maximum = \overline{\tfrac{1}{4}a^2x^2 - \tfrac{1}{4}x^4}\big|^{\frac{1}{2}}$; or $a^2x^2 - x^4 =$ a *Maximum*; the *Fluxion* of which is $=0$, i. e. $2a^2x\dot{x} - 4x^3\dot{x} = 0$; which divided by $2x\dot{x}$ makes $a^2 - 2x^2 = 0$; therefore $x^2 = \frac{1}{2}a^2$: Consequently AC=CB.

Or thus. PUT AB$=a$, AC$=x$, CB$=y$, DB$=z$; and suppose the Point b indefinitely near to B, bc parallel to B C, and B n equal and parallel to C c, i. e. let B$n = x'$, and Bb $= z'$. Now, if we consider the Increment Bb as a little right Line perpendicular to the Radius A B, the Angles n B b and CBA will be equal, and consequently the little right angled Triangle B n b will be similar to the right angled Triangle BCA: therefore AB : BC :: bB : Bn, i. e. $a : y :: z' : x'$, or $a : y :: \dot{z} : \dot{x}$, $\therefore \dot{z} = \frac{a\dot{x}}{y}$. But, when the Triangle A C B is a *Maximum*, it is plain that A B $\times \frac{1}{2}\dot{z}$ is $=$ B C $\times \dot{x}$, i. e. $\frac{1}{2}a\dot{z} = y\dot{x}$, $\therefore \dot{z} = \frac{2y\dot{x}}{a} =$ (by the above,) $\frac{a\dot{x}}{y}$: whence $2y^2\dot{x} = a^2\dot{x}$, and $2y^2 = a^2 = x^2 + y^2$. Therefore $x = y$, or A C $=$ C B; as before.

EXAMPLE X.

34. *In the right Line* AC, *which is perpendicular to the indefinite right Line* AQ, *given* $AB = a$, *and* $BC = b$: *To find the Point* P, *where the Angle* BPC *is a* Maximum.



PUT $AC = a + b = c$, and $AP = x$; then, by 47 E. 1. $BP = \overline{a^2 + x^2}|^{\frac{1}{2}}$, and $CP = \overline{c^2 + x^2}|^{\frac{1}{2}}$. Now, by Trigonometry, BP : Radius : : AP : $s\angle ABP$, i. e. (putting Rad. $= 1$,) $\overline{a^2 + x^2}|^{\frac{1}{2}} : 1 :: x : \dfrac{x}{\overline{a^2 + x^2}|^{\frac{1}{2}}} = s\angle ABP$ $= s\angle CBP$; and $CP : s\angle CBP :: CB : s\angle BPC$, i. e. $\overline{c^2 + x^2}|^{\frac{1}{2}} : \dfrac{x}{\overline{a^2 + x^2}|^{\frac{1}{2}}} :: b :$

bx

$$\frac{b x}{\overline{a^2+x^2}|^{\frac{1}{2}} \times \overline{c^2+x^2}|^{\frac{1}{2}}} = s\angle BPC = \text{a } Maximum,$$

because the said Angle is a *Maximum* by *Q*. the *Fluxion* of which is therefore $=0$; i.e.

$$\frac{b\dot{x} \times \overline{a^2+x^2} \times \overline{c^2+x^2} - bx^2\dot{x} \times \overline{c^2+x^2} - bx^2\dot{x} \times \overline{a^2+x^2}}{\overline{a^2+x^2}|^{\frac{3}{2}} \times \overline{c^2+x^2}|^{\frac{3}{2}}}$$

$=0$; which multiplied by $\overline{a^2+x^2}|^{\frac{3}{2}} \times \overline{c^2+x^2}|^{\frac{3}{2}}$, and divided by $b\,\dot{x}$, gives $\overline{a^2+x^2} \times \overline{c^2+x^2} - x^2 \times \overline{c^2+x^2} - x^2 \times \overline{a^2+x^2} = 0$, i. e. $a^2c^2 - x^4 = 0$; $\because x^4 = a^2c^2$, and $x^2 = ac$, and $x = \overline{ac}|^{\frac{1}{2}}$: Whence, by Analogy, $a : x :: x : c$; that is, A P is a geometrical Mean between the Distances A B and A C, when the Angle BPC is a *Maximum* or the greatest possible.

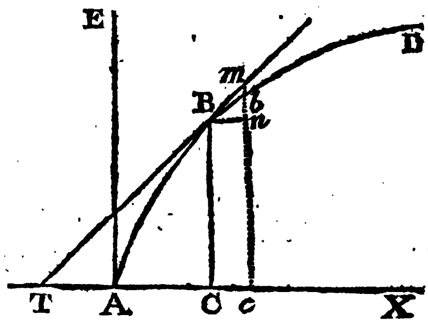
CHAP. IV.

Of drawing TANGENTS *to* CURVES.

25. A *Tangent*, is a right Line which coincides with a Curve in a Point, and there shews its Direction, that is, the Inclination it bears to the Axis, or the Angle it makes with the Ordinate.

Thus, if the right Line T B coincides with any Point B of the Curve A D, that Line

Line is a *Tangent* to it at that Point. And, becauſe no two Lines can coincide unleſs they have the ſame Direction; therefore the



Direction of the Tangent, is properly the Direction of the Curve in the Point of contact.

Now, what is requiſite, in order to draw the Tangent, is to find the right Line C T (called the Subtangent,) or the Diſtance of the Point T from the Ordinate CB, to which the Tangent muſt be drawn. To effect which,

36. Let *cb* be ſuppoſed parallel to the Ordinate CB, and B*n* equal and parallel to C*c*: then, if the Line *cb* be removed towards CB, in a parallel Motion, until it coincides with it; the Moment before its coincidence, the Triangle B*nb* will be in its *evaneſcent* State; or, which is the ſame Thing, if the ſaid

two

two Lines be ſeparated from their coincidence, the very firſt Moment of their Separation produces the ſaid Triangle in its *naſcent* State; and in that Moment, the Line *nb*, terminated by the Line B *n* at one End, and by the Curve at the other, comes indefinitely near to touch the Tangent T B produced: Conſequently, the Triangles *b n* B and B C T come then indefinitely near to ſimilarity, and may be conſidered as ſimilar: Wherefore, $bn : n\mathrm{B} :: \mathrm{BC} : \mathrm{CT}$; that is (putting Abſciſs $\mathrm{AC} = x$, Ordinate $\mathrm{CB} = y$, $\mathrm{C}c = \mathrm{B}n = x'$, and $bn = y'$,) $y' : x' :: y : \frac{x'y}{y'} = \mathrm{CT}$; or (*Article* 6.) $\dot{y} : \dot{x} :: y : \frac{\dot{x}y}{\dot{y}} = \mathrm{CT}$.

37. *Or* (to find the ſame Expreſſion for the Subtangent C T, without the help of *Increments*:) Suppoſe the indefinite right Line A E to move with a parallel and uniform Motion along the Axis A X; and, at the ſame Time, a Point to move from A along the ſaid Line AE, with ſuch Velocity, as always to be in the Curve A D: then, when the Line A E comes to the Situation C B, if the Point moving along thereon were to continue on with an uniform Motion, and with the ſame Degree of Velocity with which it

it arrives at B, it is evident, it would move along the Tangent TB produced; and therefore, when the Point A or C arrives at *c*, the Point moving along the Line A E would arrive at *m*: and, becauſe Velocity is always as the Space uniformly deſcribed in a given Time, therefore C*c* or B*n* will be as the Velocity with which the Point A moves along the Axis, *nm* as the Velocity with which the Point moves from B along the Line AE, and B*m* as the Velocity with which the Point deſcribes the Curve at B, or, moves from B to *m*: that is, C*c* or B*n* will be as the *Fluxion* of the Abſciſs A C, *nm* as the *Fluxion* of the Ordinate C B, and B*m* as the *Fluxion* of the Curve at the Point B: But the Triangles *mn*B and B C T are ſimilar; therefore, *m n* : *n*B :: BC : CT; that is, *Fluxion* of the Ordinate: *Fluxion* of the Abſciſs :: BC : CT; or (putting A C $= x$, and C B $= y$,) $\dot{y} : \dot{x} :: y : \frac{\dot{x}y}{\dot{y}} =$ CT; as before.

38. Now, this $(\frac{\dot{x}y}{\dot{y}})$ is a general Expreſſion for the Subtangent of every Curve whoſe Abſciſs is x and Ordinate y; but, as it is embaraſſed with the *Fluxions* of x and y, ſo our whole Buſineſs is to exterminate them:

To

To do which; put the Equation of the Curve into *Fluxions*; from which, or from other Properties of the Curve, find the Value of $\dot{x}$ in Terms that are all affected with $\dot{y}$, or, of $\dot{y}$ in Terms that are all affected with $\dot{x}$; then, if for $\dot{x}$ or $\dot{y}$, we ſubſtitute its Value, thus found, in this general Expreſſion $\frac{\dot{x}y}{\dot{y}}$, we ſhall have the Subtangent CT in *finite* or known Terms freed from *Fluxions*; by which the ſought Tangent T B, to a given Point in the Curve, may be eaſily drawn.

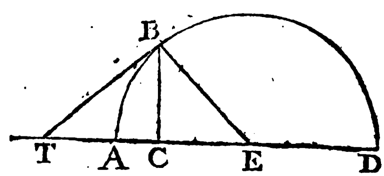
39. *Note.* WHEN $\dot{x}$ and $\dot{y}$ are negative to each other (*Art.* 19.) the general Expreſſion ($\frac{\dot{x}y}{\dot{y}}$) for the Subtangent, will be $-\frac{\dot{x}y}{\dot{y}}$; where the negative Sign only ſhews, that, the Subtangent lies on the other Side of the Ordinate, with regard to the Abſciſs x: For though $-\frac{\dot{x}y}{\dot{y}}$ may be, and really is, a negative algebraic Quantity; yet it may alſo repreſent a geometrical Quantity, which is always Affirmative: And, as every Subtangent is in its own Nature poſitive; therefore, the negative Sign, either in the general or de-

finitive Expreſſion, only ſhews where we muſt look for the Subtangent; that is, whether x, the Abſciſs, or ſome Part of it, be, (as in the affirmative Expreſſion it always is,) or be not, (as in the negative it never is,) included in the Subtangent.

EXAMPLE I.

40. *To draw a Tangent to a Circle.*

PUT the Abſciſs $AC = x$, Ordinate $CB = y$, and Radius EA or $ED = a$; then $CD =$



$2a - x$. Now, by a well known Property of Circles, $AC \times CD = \overline{CB}|^2$, i. e. $2ax - x^2 = y^2$: And this Equation put into *Fluxions*, is $2a\dot{x} - 2x\dot{x} = 2y\dot{y}$; which divided by $2a - 2x$, makes $\dot{x} = \frac{2y\dot{y}}{2a - 2x} = \frac{y\dot{y}}{a - x}$; which ſubſtituted for $\dot{x}$ in the general Expreſſion for the Subtangent (*viz.* $\frac{\dot{x}y}{\dot{y}}$, *Art.* 38.) makes the Subtangent $CT = \frac{y^2}{a - x}$ (which, by writ-

ing

ing $2ax-x^2$ for its above Value, *viz.* y^2, is) $= \frac{2ax-x^2}{a-x} = \frac{\overline{CB}|^2}{EC}$. Wherefore, if the Distance ſignified by this Expreſſion, be ſet off from the Point C, in the Diameter DA produced; we ſhall then have the Point T, to which the Tangent from the Point B muſt be drawn.

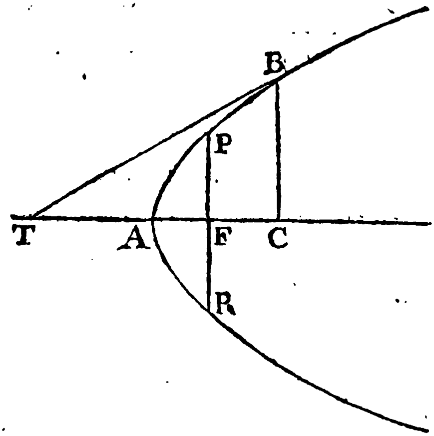
N, B. The Subtangent above, may be found otherwiſe, by the ſimilarity of Triangles only: For, the $\angle EBT$ being a right one; the right angled Triangles ECB and BCT will be ſimilar; and therefore, $EC : CB :: BC : CT = \frac{\overline{CB}|^2}{EC}$; as before.

EXAMPLE II.

41. *To draw a Tangent to the common or conic Parabola.*

SUPPOSE F to be the Focus; and PR the Parameter, which put $= a$; alſo, put the Abſciſs $AC = x$, and Ordinate $CB = y$, (B being the Point to which the Tangent TB is required to be drawn.) Then, as is well known, by the Nature of the Curve,

I 2 PR×



$PR \times AC = \overline{CB}|^2$; that is, $ax = y^2$. Now, by putting both Sides of this Equation into *Fluxions*, we ſhall have $a\dot{x} = 2y\dot{y}$; and therefore, $\dot{x} = \frac{2y\dot{y}}{a}$; which ſubſtituted for $\dot{x}$, makes the general Expreſſion for the Subtangent C T, *viz.* $\frac{\dot{x}y}{\dot{y}}$ (*See Art.* 38.) $= \frac{2y^2}{a} =$ (by ſubſtituting for y^2, its equal ax above,) $\frac{2ax}{a} = 2x$. Whence it appears, that, the Subtangent C T is double the Abſciſs AC; and conſequently, A T is = A C.

42. BUT, to draw Tangents to all ſorts of Parabolas univerſally; let the Abſciſs = x, Ordinate = y, and the Parameter = 1. Then, the Equation will be $x = y^m$, the

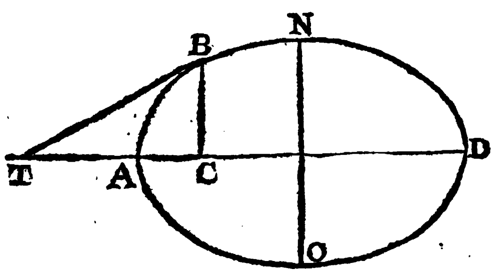
Fluxion

Fluxion of which is, $\dot{x}=my^{m-1}\dot{y}$, which Value of $\dot{x}$ being substituted for it, makes the Subtangent $\frac{\dot{x}y}{\dot{y}}=my^m=$ (by putting x for y^m its Value,) mx: So that, when $m=2$, it will be $=2x$; as before. [*See* Appendix, *Prob.* 4.]

EXAMPLE III.

43. *To draw a Tangent to an Ellipsis.*

PUT the transverse Diameter $AD=t$, Conjugate $NO=c$, Abscifs $AC=x$, and Ordinate $CB=y$. Now, by the Nature of



the Curve, $\overline{AD}|^2 : \overline{NO}|^2 :: AC\times CD : \overline{CB}|^2$; i. e. $t^2 : c^2 :: x \times \overline{t-x} : \frac{c^2}{t^2}\overline{tx-x^2}=y^2$: And, by putting each Side of this Equation into *Fluxions*, we shall have $\frac{c^2}{t^2}\overline{t\dot{x}-2x\dot{x}}=$

2yẏ;

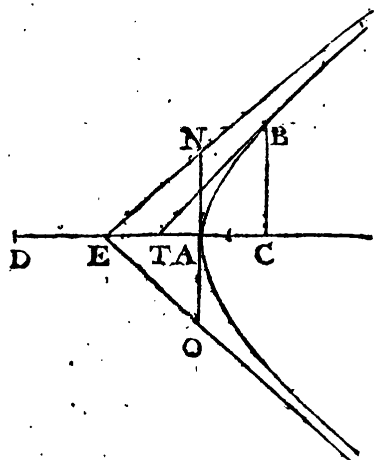
$2y\dot{y}$; and this divided by $\frac{c^2}{t^2}\overline{t-2x}$, gives $\dot{x} =$ $\frac{2t^2y\dot{y}}{c^2t-2c^2x}$, which substituted for $\dot{x}$ in $\frac{\dot{x}y}{\dot{y}}$ (the general Expression for the Subtangent; *See Article* 38.) makes the Subtangent C T $= \frac{2t^2y^2}{c^2t-2c^2x} =$ (by writing $\frac{c^2}{t^2}\ \overline{tx-x^2}$ for y^2 its equal,) $\frac{2c^2t^3x-2c^2t^2x^2}{c^2t^3-2c^2t^2x} = \frac{2tx-2x^2}{t-2x}$. Whence we may observe, that, A T ($=$ C T $-$ C A $= \frac{2tx-2x^2}{t-2x} - x$) $= \frac{tx}{t-2x}$ $=$ that Part of the Subtangent which falls *without* the Curve.

EXAMPLE IV.

44. *To draw a Tangent to an Hyperbola.*

PUT

PUT the Transverse Diameter D A $= t$, Conjugate N O $= c$, Abscis A C $= x$, and Ordinate CB$=y$. Now, by the Nature of



the Curve, $\overline{DA}|^2 : \overline{NO}|^2 :: DC \times AC : \overline{CB}|^2$, i. e. $t^2 : c^2 :: \overline{t+x} \times x : \frac{c^2}{t^2}\overline{tx+x^2} = y^2$; and, by putting both Sides of this Equation of the Curve into *Fluxions*, we shall have $\frac{c^2}{t^2} \times \overline{t\dot{x}+2x\dot{x}} = 2y\dot{y}$; therefore, by Division, $\dot{x} = \frac{2t^2 y\dot{y}}{c^2 t + 2c^2 x}$, which multiplied by $\frac{y}{\dot{y}}$, or, which is the same, substituted for $\dot{x}$ in $\frac{\dot{x}y}{\dot{y}}$, the general Expression for the Subtangent

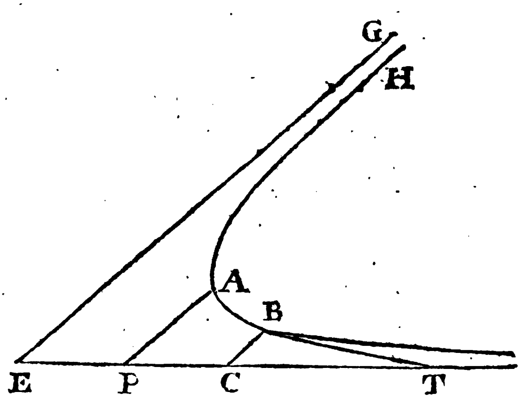
(*Art.*

(*Art.* 38.) makes the Subtangent C T $= \frac{2t^2y^2}{c^2t+2c^2x}$ (which, by writing for y^2, its equal $\frac{c^2}{t^2}\overline{tx+x^2}$, is) $= \frac{2tx+2x^2}{t+2x}$. So that, A T, that Part of the Subtangent *without* the Curve, is $= \frac{2tx+2x^2}{t+2x} - x = \frac{tx}{t+2x}$.

EXAMPLE V.

45. *To draw a Tangent to an Hyperbola between its Asymptotes; that is, taking one of its Asymptotes for an Axis.*

LET E G and E T be the Asymptotes of the Hyperbola H A B, whose Vertex is



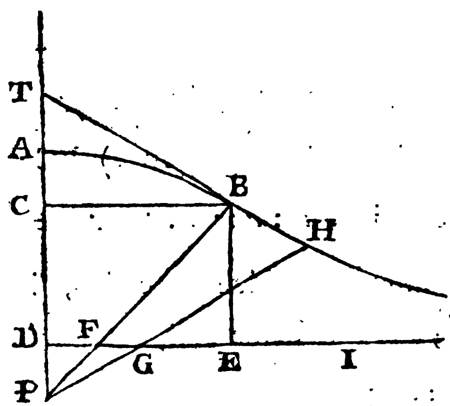
A: draw AP and BC parallel to the Asymptote E G: then will A P be the Parameter, and

and equal to P E, which put $= a$; E C an Abſciſs, which put $= x$; and C B an Ordinate, which put $= y$. Now, becauſe when x increaſes, y decreaſes; therefore $\dot{x}$ and $\dot{y}$ are negative to each other; and the general Expreſſion for the Subtangent is $-\frac{\dot{x}y}{\dot{y}}$; where the negative Sign ſhews, that, the Point T lies on the other Side of the Ordinate C B with regard to E, (*Art.* 39.) By the Property of the Curve (See *Stone's De L'Hoſpital's Conic Sections*, Art. 101.) EC : EP :: PA : CB, i. e. $x : a :: a : y$; $\because x = \frac{a^2}{y}$; the *Fluxion* of which Equation, is $\dot{x} = \frac{-a^2\dot{y}}{y^2}$; and this Value of $\dot{x}$, being ſubſtituted for it in the above general Expreſſion for the Subtangent, *viz.* $-\frac{\dot{x}y}{\dot{y}}$, makes the Subtangent $CT = \frac{a^2y\dot{y}}{y^2\dot{y}} = \frac{a^2}{y} =$ (by writing xy for its equal a^2,) $\frac{xy}{y} = x$. So that, CT muſt be $=$ C E.

EXAMPLE VI.

46. *To draw a Tangent to the Conchoid of* Nicomedes. *

LET fall the Perpendicular BE on the Asymptote DI; and draw BC equal and



parallel to ED. Put $PD=a$, $DA=b=FB$, $DC=x=EB$, and $CB=y=DE$. Then, by

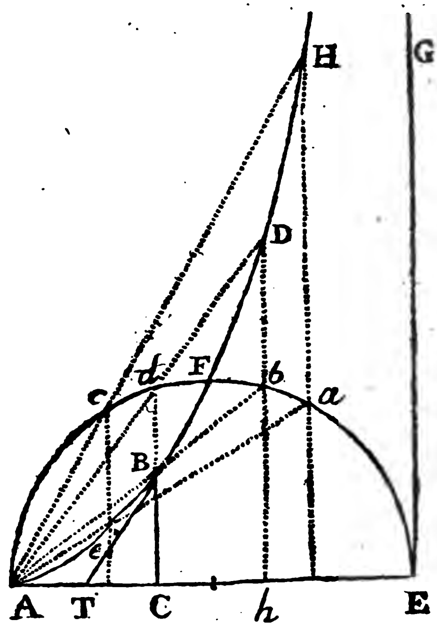
* This Curve is thus generated: — From a fix'd Point P, which is called the Pole of the Conchoid, let an infinite Number of right Lines PA, PB, PH, be drawn, cutting the right Line DI, which is an Asymptote to the Curve; and let the Distances DA, FB, GH, be made equal to each other, and a Line drawn through the Points A, B, H, &c. then will this Line be a Curve, called by its Inventor *Nicomedes*, a *Conchoid*.

by 47 E. 1. $\overline{\overline{FB}|^2 - \overline{BE}|^2}|^{\frac{1}{2}} = EF$, that is, $\overline{b^2 - x^2}|^{\frac{1}{2}} = EF$. But, the Δs CPB and BEF are similar; therefore, BE : EF : : PC : CB, i. e. $x : \overline{b^2 - x^2}|^{\frac{1}{2}} : : a + x : y = \frac{a+x}{x} \times \overline{b^2 - x^2}|^{\frac{1}{2}}$; which is the Equation of the Curve; and this in *Fluxions* is $\dot{y} = \frac{x\dot{x} - \overline{a+x}\dot{x}}{x^2} \times \overline{b^2 - x^2}|^{\frac{1}{2}} - \frac{1}{2} \times \overline{b^2 - x^2}|^{-\frac{1}{2}} 2x\dot{x} \times \frac{a+x}{x} = \frac{-a\dot{x}}{x^2} \times \overline{b^2 - x^2}|^{\frac{1}{2}} - \frac{x\dot{x}}{\overline{b^2 - x^2}|^{\frac{1}{2}}} \times \frac{a+x}{x} = \frac{-ab^2\dot{x} - x^3\dot{x}}{x^2 \times \overline{b^2 - x^2}|^{\frac{1}{2}}}$; which substituted for $\dot{y}$, in $-\frac{\dot{x}y}{\dot{y}}$, the general Expression for the Subtangent when $\dot{x}$ and $\dot{y}$ are negative to each other (*Art.* 39.) makes the Subtangent C T $= \frac{yx^2 \times \overline{b^2 - x^2}|^{\frac{1}{2}}}{ab^2 + x^3}$ (which, by substituting for y its equal in the above Equation of the Curve, is) $= \frac{\overline{a+x} \times \overline{b^2 - x^2} \times x^4}{ab^2x + x^4} = \frac{\overline{a+x} \times \overline{b^2x - x^3}}{ab^2 + x^3}$.

EXAMPLE VII.

47. *To draw a Tangent to the Cissoid of* Diocles. *

LET ABH be the Cissoid, whose gene-



rating Semicircle is A F E, and Asymptote E G.

* This Curve is thus generated: — In the Semicircle A F E, make any Arches E *a*, and A *c*, E *b*, and A *d*, equal to one another; and through the Points *a*, *b*, *d*, *c*, let right Lines be drawn perpendicular to the Diameter A E, and transverse Lines from the Point A; then from A, through the Points of Intersection *e*, B, D, &c. draw a Line A*e*BD, &c. and it will be a Curve, called by its Inventor *Diocles*. a *Cissoid*.

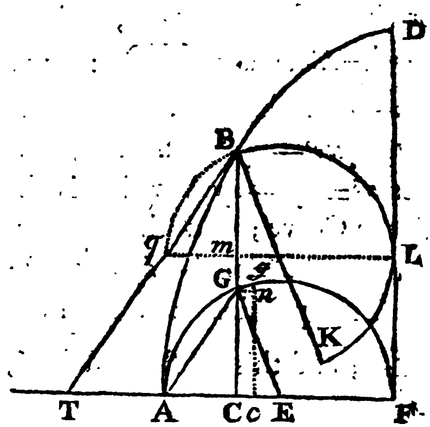
E G. Put the Diameter A E $= a$, Abſciſs AC $= x$, and Ordinate C B $= y$. Now, by the Nature of the Curve, $dC : CA :: AC : CB$; (For, ſince by the Generation of the Curve, the Arches E b and A d muſt be equal; therefore $bb = dC$, and $bE = CA$; and, becauſe the Δs bbA and BCA are alike, therefore $Ab : bb :: AC : CB$; but, by the Property of Circles, $Ab : bb :: bb : bE$; $\because$ $bb : bE :: AC : CB$, i. e. $dC : CA :: AC : CB$;) or (becauſe $CE = a - x$, and $\overline{AC \times CE}|^{\frac{1}{2}} = \overline{ax - x^2}|^{\frac{1}{2}} = Cd$,) as $\overline{ax - x^2}|^{\frac{1}{2}} : x :: x : y$; $\because$ $x^2 = y \times \overline{ax - x^2}|^{\frac{1}{2}}$; the Square of which Equation is $x^4 = axy^2 - x^2y^2$; and this, divided by x, makes $x^3 = ay^2 - xy^2$; which is the Equation of the Curve; and the *Fluxion* of this Equation is $3x^2\dot{x} = 2ay\dot{y} - \dot{x}y^2 - 2xy\dot{y}$; therefore, by Tranſpoſition and Diviſion, $\dot{x} = \frac{2ay\dot{y} - 2xy\dot{y}}{3x^2 + y^2}$; and this ſubſtituted for $\dot{x}$, makes the Subtangent CT $\left(= \frac{\dot{x}y}{\dot{y}}, \text{ Article } 38.\right) = \frac{2ay^2 - 2xy^2}{3x^2 + y^2} =$ (by ſubſtituting for y^2, its equal $\frac{x^3}{a - x}$,) $\frac{2ax - 2x^2}{3a - 2x}$. Whence we may obſerve, that AT, the Difference between the Abſciſs AC

AC and Subtangent TC, is $=\frac{ax}{3a-2x}$; the Abſciſs AC being $=\frac{3ax-2x^2}{3a-2x}$.

EXAMPLE VIII.

48. *To draw a Tangent to the Cycloid.* *

PUT EA the Radius of the generating Circle $=a$, Abſciſs AC $=x$, Ordinate CB $=y$,



CG $=s$, and the Arch AG $=z$. Now, by the

* This Curve is thus generated: — Let a Circle, or Wheel, roll along upon a right Line until it performs one Revolution, that is, until it meaſures out a right Line equal to its Circumference; then, that Point in the Circle which firſt touched the right Line, will deſcribe the Curve called a *Cycloid*.

the Nature of the Curve, CB=CG + Arch GA; that is, $y=s+z$; (For, when the Semicircle AGF, generating the Semicycloid ABD, is in the Position BLK, the Arch BL or GF must be equal to LD, and LK or Bq or GA=LF or mC; but Cm=GB; therefore Arch GA = GB; and therefore, &c.) and the *Fluxion* of this Equation of the Curve is $\dot{y}=\dot{s}+\dot{z}$. Let cg be supposed indefinitely near and parallel to CG, and Gn equal and parallel to Cc; that is, let Gg=z', and gn =s', and nG=Cc=x'; then, supposing Gg to be a little right Line perpendicular to GE, the ∠s, gGn and EGC will be equal; and therefore the right angled △s, gnG and GCE are alike; and consequently, $gn : n$G :: EC : CG; that is, $s' : x' :: a-x : s$; ∵ $s' = \frac{a-x}{s}x'$, or, (*Art.* 6.) $\dot{s} = \frac{a-x}{s}\dot{x}$; And again, gG : Gn :: EG : GC; that is, $z' : x' :: a : s$; ∵ $z' = \frac{a}{s}x'$, or, $\dot{z} = \frac{a}{s}\dot{x}$. Now, by substituting $\frac{a-x}{s}\dot{x}$ for $\dot{s}$, and $\frac{a}{s}\dot{x}$ for $\dot{z}$, in the above *Fluxion* of the Equation of the Curve, we have $\dot{y} = \frac{a-x}{s}\dot{x} + \frac{a}{s}\dot{x} = \frac{2a-x}{s}\dot{x}$; and this substituted for $\dot{y}$, makes the Subtangent

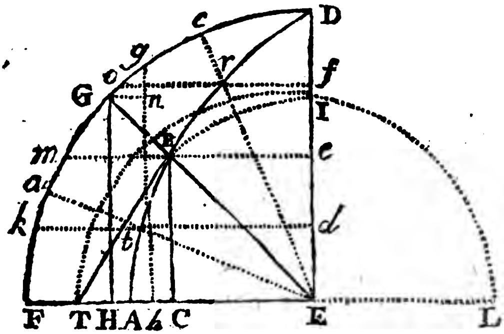
gent CT ($=\frac{\dot{x}y}{\dot{y}}$, *Art.* 38.) $=\frac{sy}{2a-x}$.
Whence we may obſerve, by Analogy, $2a-x$: s : : y : CT ; i. e. FC : CG : : BC : CT : but, by the Property of Circles, FC : CG : : GC : CA ; ∵ GC : CA : : BC : CT : Conſequently, the $\triangle$s, GCA and BCT are alike, and BT is parallel to the Chord GA.

EXAMPLE IX.

49. *To draw a Tangent to the Quadratrix of* Dinoſtratus. *

PUT EF, or ED, the Radius of the generating Quadrant $=a$; EA the Baſe of the Quadratrix $=b$; Quadrantal Arch FD $=c$; Abſciſs

* This Curve is thus generated: — Let EFD be a Quadrant of a Circle, whoſe Radius ED divide into any Number of equal Parts (as E*d*, *de*, *ef*, *f*D ;) and from the Points of Diviſion (*d*, *e*, *f*,) draw right Lines (*dk*, *em*, *fo*,) parallel to each other, and to the Radius EF : alſo, divide the Arch FD into the ſame Number of equal Parts (F*a*, *a*G, G*c*, *c*D,) as you do the Radius ED : and to theſe Points of Diviſion (*a*, G, *c*,) draw Radii (E*a*, EG, E*c*,) from the Point or Center E. Then, a Line drawn from E through the Points of Interſection (*r*, B, *t*, &c.) will be a Curve, called a *Quadratrix*; the Invention of which is imputed to *Dinoſtratus*.



Abſciſs A C $=x$; Ordinate C B $=y$; Arch FG $=z$; and the Sine GH $=s$. Now, by the Nature of the Curve, as Arch FD : Rad. DE : : Arch FG : Ordinate BC; (which is evident from the Generation of the Curve;) that is, $c : a :: z : y$; $\because az = cy$, and $z = \frac{cy}{a}$; the *Fluxion* of which, is $\dot{z} = \frac{c\dot{y}}{a}$. By Sim. $\triangle$s, BC : CE : : GH : HE, i. e. $y : b - x :: s : \frac{sb - sx}{y} =$ HE. Let gb be conceived indefinitely near and parallel to GH, and Gn equal and parallel to Hb; i. e. let $ng = s'$, and $gG = z'$; then, Gg being conſidered as coinciding with a Tangent to the Point G, the $\triangle$s E H G and gnG will be alike; and therefore, H E : E G : : $ng : g$G, that is, $\frac{sb - sx}{y} : a :: s' : z' = \frac{as'y}{sb - sx}$, or, (*Art.* 6.)

 $\dot{z} =$

$\dot{z} = \frac{a\dot{s}y}{sb - sx}$. Hence we have $\frac{a\dot{s}y}{sb - sx} = \frac{c\dot{y}}{a}$; which gives $\dot{s} = \frac{sbc\dot{y} - sxc\dot{y}}{a^2y}$. Now, by 47 E. 1. $\overline{EC^2 + CB^2}|^{\frac{1}{2}} = BE$, i. e. $\overline{\overline{b-x}|^2 + y^2}|^{\frac{1}{2}} = BE$; and, by Sim. Δs, EB : BC : : EG : GH, that is, $\overline{\overline{b-x}|^2 + y^2}|^{\frac{1}{2}} : y : : a : s = \frac{ay}{\overline{\overline{b-x}|^2 + y^2}|^{\frac{1}{2}}}$, or $s = \frac{ay}{\overline{b^2 - 2bx + x^2 + y^2}|^{\frac{1}{2}}}$; in

$$Flux.\ \dot{s} = \frac{a\dot{y} \times \overline{\overline{b-x}|^2 + y^2}|^{\frac{1}{2}} + \frac{b\dot{x}ay - x\dot{x}ay - y\dot{y}ay}{\overline{\overline{b-x}|^2 + y^2}|^{\frac{1}{2}}}}{\overline{b-x}|^2 + y^2}$$

$= \frac{a\dot{y} \times \overline{b-x}|^2 + b\dot{x}ay - x\dot{x}ay}{\overline{\overline{b-x}|^2 + y^2}|^{\frac{3}{2}}} =$ (becauſe $\dot{s}$ by the above is $=$) $\frac{sbc\dot{y} - sxc\dot{y}}{a^2y}$; and this Equation, by Multiplication and Tranſpoſition, produces $a^3by^2\dot{x} - a^3y^2x\dot{x} = \overline{bcs\dot{y} - csx\dot{y}} \times \overline{\overline{b-x}|^2 + y^2}|^{\frac{3}{2}} - a^3y\dot{y} \times \overline{b-x}|^2$; and this divided by $b-x$, makes $a^3y^2\dot{x} = cs\dot{y} \times \overline{\overline{b-x}|^2 + y^2}|^{\frac{3}{2}} - a^3y\dot{y} \times \overline{b-x}$ (or, by ſubſtituting for s, its above Value $\frac{ay}{\overline{\overline{b-x}|^2 + y^2}|^{\frac{1}{2}}}$,) $= acy\dot{y} \times \overline{\overline{b-x}|^2 + y^2} - a^3y\dot{y} \times \overline{b-x}$. But, by a Property of the Curve, as F D : D E : : E F : E A;

(For,

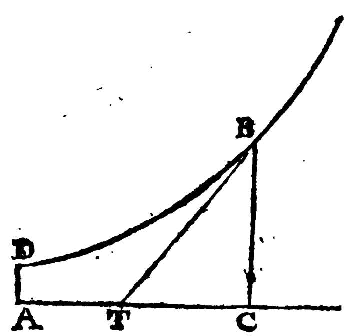
(For, when the Arch F G is indefinitely ſmall, it will nearly coincide with, or be equal to its Sine GH, and one may be taken for the other; and then, EA may be taken for E C, and E F for E H; and therefore, then, by the above Nature of the Curve, as Quadrantal Arch F D : Rad. D E : : indefinitely ſmall Sine GH : indefinitely ſmall Ordinate BC; but, by Sim. $\triangle$s, GH : BC : : (HE, i. e.) EF : (CE, i. e.) EA : Conſequently, Arch FD : Rad. DE : : EF : EA :) i. e. as $c : a :: a : b$; $\because bc = a^2$; and by ſubſtituting bc for a^2, we have $abcy^2\dot{x} = acy\dot{y} \times \overline{\overline{b-x}|^2 + y^2} - abcy\dot{y} \times \overline{b-x}$; and this Equation divided by acy, makes $by\dot{x} = \dot{y} \times \overline{\overline{b-x}|^2 + y^2} - b\dot{y} \times \overline{b-x}$; $\because \dot{x} = \dot{y} \times \frac{\overline{\overline{b-x}|^2 + y^2} - b \times \overline{b-x}}{by}$; which ſubſtituted for $\dot{x}$, makes the Subtangent C T $\left(= \frac{\dot{x}y}{\dot{y}}\right.$, *Article* 38.) $= \frac{\overline{b-x}|^2 + y^2}{b} - \overline{b-x} = \frac{\overline{EB}|^2}{EA} - E\,C$; which gives the following geometrical Conſtruction, *viz.* If B be the Point to which the Tangent is to be drawn; make EI=EB, and EL=EA; and let the Semicircle LIT be drawn through the Points L and I; then T

is the Point through which the Tangent must pass: For, by the Property of Circles, $LE \times ET = \overline{EI}|^2 =$ (by Construction) $\overline{EB}|^2$; i. e. $EA \times \overline{EC+CT} = \overline{EB}|^2$; $\therefore \frac{\overline{EB}|^2}{EA} - EC = CT$.

EXAMPLE X.

50. *To draw a Tangent to the exponential Curve* DB, *whose Equation (putting* $AC = x$, $CB = y$, *and* $a =$ *a given Quantity* more *than* 1,*) is* $a^x = y$.

PUT $A =$ the hyperbolic Logarithm of a, and $Y =$ the hyperbolic Logarithm of y; then, by the Nature of Logarithms, $xA = Y$; in *Fluxions*, $\dot{x}A = \dot{Y} =$ (See *Article* 17.)



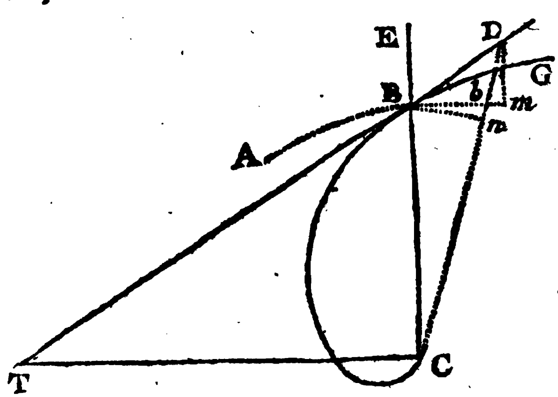
$\frac{\dot{y}}{y}$; which divided by A, makes $\dot{x} = \frac{\dot{y}}{Ay}$; and this substituted for $\dot{x}$, makes the Subtangent

CT

CT ($=\frac{\dot{x}y}{\dot{y}}$, *Art.* 38.) $=\frac{y}{Ay}=\frac{1}{A}$. Whence we may obſerve, that, the Subtangent being an invariable Quantity, the Curve DB is the logarithmic Curve. (See *Art.* 77 *and* 78.)

51. HITHERTO we have treated only of Curves referred to an Axis; or, of thoſe whoſe Ordinates are parallel to one another: We therefore now proceed to the drawing of Tangents to *Spirals*; or to thoſe Curves, all of whoſe Ordinates iſſue from one and the ſame fixt Point. Where *Note*, the Ordinate and Subtangent are always perpendicular to each other.

52. SUPPOSE C*b* indefinitely near to CB; that is, let the ∠BC*b* be ſuppoſed indefinitely



ſmall; and with the Ordinate CB, as a Radius,

dius, let the indefinitely ſmall Arch B*n* be deſcribed; which being conſidered as a little right Line perpendicular to C*b*, and the indefinitely ſmall Part of the Curve B*b* as coinciding with a Tangent to it at the Point B; then, becauſe the $\angle BCb$ is indefinitely ſmall, the $\triangle$s *bn*B and BCT will be indefinitely near to ſimilarity; that is, in the very firſt Moment of the Exiſtance of the $\triangle bnB$, it will be ſimilar to the $\triangle BCT$: Therefore, if we put the Ordinate $CB=y$, $Bn=x'$, and $nb=y'$; we ſhall have $y' : x' :: y : CT$; that is (*Art.* 6.) $\dot{y} : \dot{x} :: y : CT$; $\therefore CT=\frac{\dot{x}y}{\dot{y}}$; which is the ſame general Expreſſion for the Subtangent as that before found for Curves referred to an Axis, (*Art.* 38.)

53. BUT, this ſame general Expreſſion, may be obtained without the help of *Increments*, or the ſuppoſition of indefinitely ſmall Quantities, thus,——Let the indefinite right Line CE turn, like the Radius of a Circle, round the fixt Point or Center C with an uniform Motion; and, at the ſame Time, let a Point moving along thereon, generate, or move with ſuch Degrees of Velocity as always to be in the Curve CBG, to which ſuppoſe TBD a Tangent at the Point B; alſo,

ſo, let B*m* be a Tangent to the circular Arch AB*n* deſcribed with the Ordinate CB as a Radius. Now, when the Point generating the Curve CBG arrives at B, if it was to continue on in the ſame Direction, with an uniform Motion, and the ſame Degree of Velocity that it arrives thereat, it would move along the Tangent T B produced, or right Line BD, which would always be as the *Fluxion* of the Spiral at the Point B: So likewiſe, if we ſuppoſe a Point to move from B, in the ſame Direction, and with the ſame uniform Motion, that the Point generating the circular Arch A B *n* arrives at B, it would move along the Tangent or right Line B *m* perpendicular to the Ordinate or Radius CB, which would always be as the *Fluxion* of the ſaid Arch at B: And ſince the Direction of the Point moving from C to E, is always perpendicular to that of the Point generating the circular Arch A *n*; therefore, when the Point moving from B to D arrives at D, if D*m* be parallel to E C, the Point moving along B*m* will be arrived at *m*; and *m*D will be as the *Fluxion* of the Ordinate at the Point B. Hence, becauſe the $\triangle s$ D *m* B and BCT are ſimilar, D*m* : *m*B : : BC : CT; that is, the *Fluxion* of the Ordi-

nate

nate CB : is to the *Fluxion* of the Arch AB : : as the Ordinate BC : is to the Subtangent CT; or (putting Arch AB $=x$, and Ordinate CB $=y$;) $\dot{y}:\dot{x}::y:\frac{\dot{x}y}{\dot{y}}=$ CT; as before.

Example I.

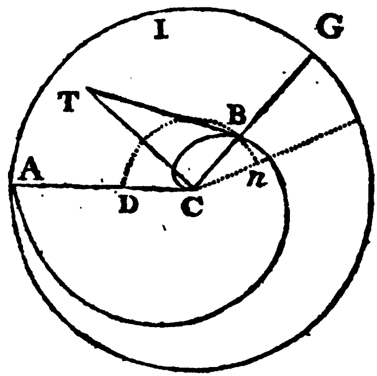
54. *To draw a Tangent to the Spiral of* Archimedes. *

Put the Circumference of the generating Circle $=a$, and its Radius CA $=b$; Ordinate CB $=y$, Arch AIG $=z$; and with the Ordinate CB, as a Radius, let the circular Arch B*n* be described, which put $=x$. Now, by the Nature of the Curve, $a:b::z:y$, or

---

* This Curve is thus generated:—With the Radius CA, let the Circle AGA be described with an equable Motion, or the Point A describe equal Arches in equal Terms; and at the same Moment of Time that the Point A begins to generate the Circle, let another Point begin to move along the said Radius from C towards A, and to pass over it with an uniform Motion, and such Velocity, that it may arrive at A at the very same Moment of Time that the said Radius shall have described the Circle, or come to be in its first Situation. Then will the Point moving along the Radius CA, generate, or describe, the Curve CB*b*A, called a *Spiral*, the invention of which is attributed to *Archimedes*.

or (*Article* 8.) $a : b :: \dot{z} : \dot{y}$; $\because \dot{z} = \frac{a\dot{y}}{b}$. But, it is evident, the Velocity of the Point



generating the Circle, is to the Velocity with which the Point B generates the Arch B*n*, as CG to CB; i. e. $\dot{z} : \dot{x} :: b : y$; $\because \dot{z} = \frac{b\dot{x}}{y}$. Hence we have $\frac{b\dot{x}}{y} = \frac{a\dot{y}}{b}$; therefore $b^2\dot{x} = ay\dot{y}$, and $\dot{x} = \frac{ay\dot{y}}{b^2}$; which substituted for $\dot{x}$ in the general Expression for the Subtangent $\frac{\dot{x}y}{\dot{y}}$ (*Article* 52.) makes the Subtangent CT $= \frac{ay^2}{b^2} =$ (because by the above $a : b :: z : y$, or $ay = bz$,) $\frac{byz}{b^2} = \frac{yz}{b}$. Which

M gives

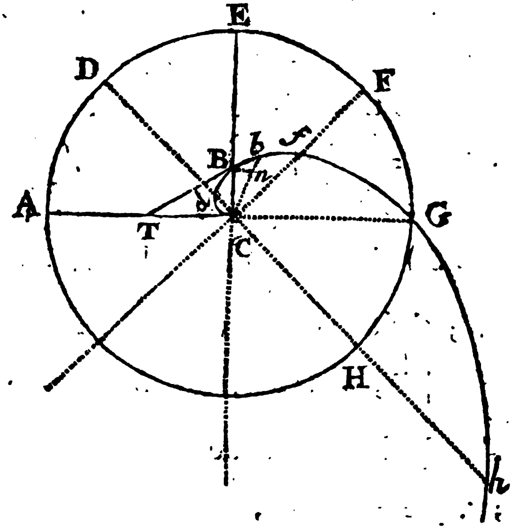
gives the following Constuction, *viz.* With the Ordinate C B, as a Radius, describe the circular Arch BD; and make CT perpendicular to C G, and equal to the Arch B D: then will T be the Point through which the Tangent B T produced must pass. For, then, the Sectors CGA and CBD will be similar; and consequently, CG : GA : : CB : BD; i. e. $b : z : : y : BD = \frac{yz}{b} = CT$.

Example II.

55. *To draw a Tangent to the Logarithmic Spiral.* *

Put the Curve C *d* B $= z$, and its corresponding Ordinate B C $= y$. Now, if we suppose the Points B, *f*, G, *h*, &c. indefinitely near to each other, the little Parts of the

* This Curve is thus generated:—In the Circle AFH, let the Arches AD, DE, EF, FG, GH, *&c.* be made equal to each other; and the righr Lines C*d*, CB, C*f*, CG, C*h*, *&c.* in geometrical Proportion continued: then, a Line drawn from the Center C through the Points *d*, B, *f*, G, *h*, *&c.* will be the Curve called a *Logarithmic Spiral*; which Name is given to it, because the Arches AD, AE, AF, *&c.* will be the Logarithms of the corresponding Ordinates C*d*, CB, C*f*, *&c.*



the Curve Bf, fG, &c. may be considered as indefinitely ſmall right Lines; and therefore (becauſe by the Generation of the Curve, CB : Cf : : Cf : CG, and the Angles BCf and fCG are equal,) the little △s CBf and CfG will be ſimilar: Conſequently, the Angle made by the Curve and Ordinate is always the ſame; and the Ordinate and Curve are in a given or fixt Proportion, which ſuppoſe to be as a to b; that is, $y : z :: a : b$; therefore (*Art.* 8.) $\dot{y} : \dot{z} :: a : b$; $\because \dot{z} = \frac{b\dot{y}}{a}$. Let C$b$ be ſuppoſed indefinitely near to CB; and the little Arch Bn (deſcribed with the Radius

 CB)

CB) as a little right Line perpendicular to Cb; then, Bb being confidered as coinciding with a Tangent to the Point B, we fhall have, by 47 E. 1. $Bb = \overline{\overline{Bn}|^2 + \overline{nb}|^2}|^{\frac{1}{2}}$; that is, (putting $Bb = z'$, $Bn = x'$, and $nb = y'$,) $z' = \overline{x'^2 + y'^2}|^{\frac{1}{2}}$; or

(*Art.* 6.) $\dot{z} = \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}}$. Hence, $\frac{b\dot{y}}{a} = \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}}$; the Square of which Equation, is $\frac{b^2\dot{y}^2}{a^2} = \dot{x}^2 + \dot{y}^2$; and this, by Multiplication and Tranfpofition, produces $b^2\dot{y}^2 - a^2\dot{y}^2 = a^2\dot{x}^2$; $\therefore \dot{y}^2 = \frac{a^2\dot{x}^2}{b^2 - a^2}$, and $\dot{y} = \frac{a\dot{x}}{\sqrt{b^2 - a^2}}$; which being fubftituted for $\dot{y}$ in the general Expreffion for the Subtangent, *viz.* $\frac{\dot{x}y}{\dot{y}}$ (*Article* 52.) makes the fought Subtangent $CT = \frac{\dot{x}y\sqrt{b^2 - a^2}}{a\dot{x}} = y \times \frac{\sqrt{b^2 - a^2}}{a}$. And hence we may obferve, that, (becaufe by 47 E. 1. $\overline{TB}|^2 = \overline{BC}|^2 + \overline{CT}|^2$, i. e. $\overline{TB}|^2 = y^2 + y^2 \times \frac{b^2 - a^2}{a^2} = \frac{b^2y^2}{a^2}$,) the Tangent TB is $= \frac{by}{a} = y \times \frac{b}{a}$; and confequently, the Tangent and Subtangent are to each other as b to $\sqrt{b^2 - a^2}$.

CHAP.

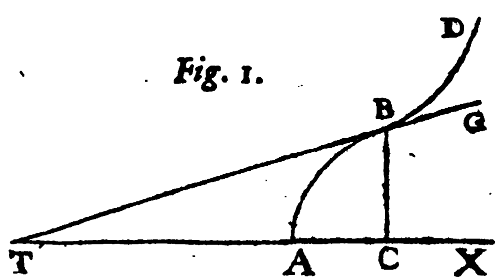
CHAP. V.

Of finding the Points of Inflection, *or of* contrary Flexure, *in* Curves.

56. WHEN a Curve from being Concave, becomes Convex towards its Axis; or, from being Convex, becomes Concave; then, that Point in it, where the Change is made, or that which separates the Convex from the Concave Part, is called the Point of *Inflection*, or of *contrary Flexure*. So that, if at the Point of *Inflection*, a Tangent be drawn, it will there cut the Curve.

Thus, in *Fig.* 1. if AB be Concave, and BD Convex; or, in *Fig.* 2. if AB be Con-

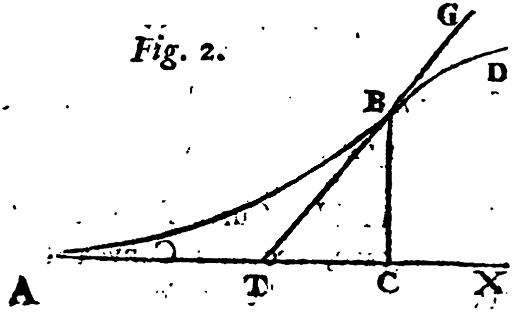
Fig. 1.



vex, and DB Concave towards the Axis AX; then B is the Point of *Inflection*; where, the Tangent TBG cuts the Curve.

57. Now,

Fig. 2.



57. Now, it is evident, that, in any Curve, in Order to determine whether the Abſciſs or Ordinate flows with an accelerated, uniform, or retarded Motion, and the Quantity of it, or the Value of its *Second Fluxion*, it is neceſſary that one of them be made to increaſe with a given uniform Motion, with which the Velocity, or ſwiftneſs of Increaſe, of the other may be always compared. And therefore, if we ſuppoſe the *Abſciſs* (as being the moſt Natural) always to increaſe with one and the ſame given Degree of Velocity, or equal Parts of it to be deſcribed in equal Times; then, becauſe the Direction of the Curve from A to B (*Fig.* 1.) or from B to D (*Fig.* 2.) continually approaches nearer to a Parallelism with the Axis; it is evident, the Ordinate between theſe Points, muſt flow with a Motion continually retarded: And, becauſe

because the Direction of the Curve from B to D (*Fig.* 1.) or from A to B (*Fig.* 2.) approaches continually nearer to a Perpendicular to the Axis, therefore, between these Points, the Ordinate must flow, or increase, with a Motion continually accelerated: Consequently, at the Point of *Inflection* B, the Ordinate will flow with a Motion neither accelerated nor retarded, and therefore, with an uniform Motion: that is, at the Point of *Inflection* B, the *Second Fluxions* of the Abscis and Ordinate will each of them be $=0$. —Which is also plain, since the *Fluxion* of the Ordinate, at the Point B, must be a *Maximum* or a *Minimum*; in either Case its *Second Fluxion* is $=0$.

58. Now, to find the *Second Fluxion* of an Equation, every *First Fluxion* in it, not supposed *invariable*, must be considered as a distinct *variable Quantity*; and then the Equation put into *Fluxions* by the *Rules* laid down in *Chap.* II.

Thus, the *First Fluxion* of the Equation $ax=y^2$, being $a\dot{x}=2y\dot{y}$; the *Second Fluxion* of it, *viz.* the *Fluxion* of $a\dot{x}=2y\dot{y}$, will be $a\ddot{x}=2\dot{y}\dot{y}+2y\ddot{y}$; that is, $a\ddot{x}=2\dot{y}^2+2y\ddot{y}$. So likewise, the *Fluxion* of $a\dot{x}-x\dot{x}=y\dot{y}$, is $a\ddot{x}-\dot{x}^2-x\ddot{x}=\dot{y}^2+y\ddot{y}$. — But, since either $\dot{x}$ or

or $\dot{y}$ may be supposed *invariable*, (i. e. either x or y to increase with an uniform Motion;) therefore, by making $\dot{x}$ invariable, or $\ddot{x}=0$, the first of above *Second Fluxional* Equations, *viz.* $a\ddot{x}=2\dot{y}^2+2y\ddot{y}$, will become $0=2\dot{y}^2+2y\ddot{y}$; or, if we make $\dot{y}$ invariable, it will be $a\ddot{x}=2\dot{y}^2$. And, if we make $\dot{x}$ invariable, the last of the said Equations, *viz.* $a\ddot{x}-\dot{x}^2-x\ddot{x}=\dot{y}^2+y\ddot{y}$, will be $-\dot{x}^2=\dot{y}^2+y\ddot{y}$; or, if $\dot{y}$ be supposed invariable, it will become $a\ddot{x}-\dot{x}^2-x\ddot{x}=\dot{y}^2$. — Also, the *Fluxion* of $z=x+\frac{y\dot{y}}{\dot{x}}$, (supposing $\dot{x}$ invariable,) is $\dot{z}=\dot{x}+\frac{\dot{y}^2+y\ddot{y}}{\dot{x}}$: Or, (supposing $\dot{y}$ invariable,) $\dot{z}=\dot{x}+\frac{\dot{y}^2\dot{x}-\ddot{x}y\dot{y}}{\dot{x}^2}$.

The *Third* and *Fourth Fluxions*, &c. are found in the same Manner; due regard being had to such *Fluxions* as are supposed *invariable*.

59. HENCE, to find the Point of *Inflection* B: Put the Equation of the Curve (where the Abscis A C is $=x$, or $a\pm x$, and the Ordinate C B $=y$,) into *Fluxions*; from which, find the Value of $\dot{x}$ or $\dot{y}$; and put this $\dot{x}$ or $\dot{y}$, and its Value, into *Fluxions* again, making both $\ddot{x}$ and $\ddot{y}=0$: then, by expunging

ing the reſt of the *fluxional* Quantities, you may have x or y, at the Point of *Inflection* ſought, determined.

60. BUT, the Point of *Inflection* may be found without the help of *Second Fluxions:* For, if T B be a Tangent to it, it is evident, that A T, the Difference between the Subtangent and Abſciſs, will be a *Maximum:* And therefore, to find the Point of *Inflection*, we need only find a definitive Expreſſion for the Subtangent, by the preceding Chapter; and the Difference between it and the Abſciſs: and then make the *Fluxion* of this Difference $=$ o.——This we ſhall illuſtrate in the following Example.

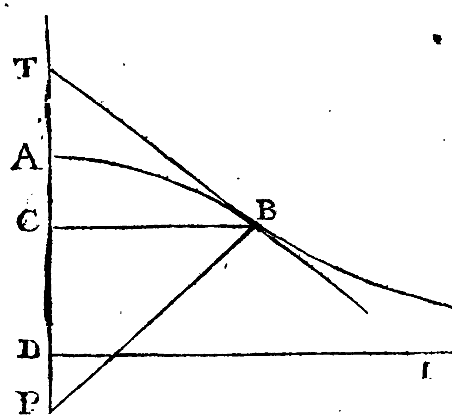
EXAMPLE I.

61. *To find the Point of Inflection in the Conchoid of* Nicomedes; *or, the Point where it begins to be Convex towards the Axis* A D.*

N PUT

* At A, the Curve is Concave towards D I; but it cannot poſſibly continue ſo long: for, if it did, the farther it proceeded, the more it would incline, or tend, towards the ſaid Line, and ſo interſect it; whereas, in fact, the ſaid Line D I is an Aſymptote to the ſaid Curve, as appears from its Generation, *Art.* 46. *Note.*

PUT PD $=a$, DA $=b$, DC $=x$, and CB $=y$; then, by *Article* 46. the *Fluxion* of the Equation of the Curve will be $\dot{y}=\frac{-ab^2\dot{x}-x^3\dot{x}}{x^2\times\overline{b^2-x^2}|^{\frac{1}{2}}}$: Suppose B to be the Point



of Inflection; then, the *Fluxion* of this again, (because, x being supposed to flow with an equable Motion, the *Second Fluxions* of x and y are each of them $=0$,) will be $\dot{o}=$

$$\frac{-3x^4\dot{x}^2\times\overline{b^2-x^2}|^{\frac{1}{2}}-:2x\dot{x}\times\overline{b^2-x^2}|^{\frac{1}{2}}-\frac{x^3\dot{x}}{\overline{b^2-x^2}|^{\frac{1}{2}}}\times-ab^2\dot{x}-x^3\dot{x}}{x^4\times\overline{b^2-x^2}}$$

(*See Articles* 13. and 58.) that is, $o=\frac{2ab^4x-3ab^2x^3-b^2x^4}{x^4\times\overline{b^2-x^2}|^{\frac{3}{2}}}\dot{x}^2$; which multiplied by $x^4.\overline{b^2-x^2}|^{\frac{3}{2}}$ and divided by $b^2x\dot{x}^2$, gives $o=2ab^2-3ax^2-x^3$, or, $x^3+3ax^2-2ab^2$

$2ab^2 = 0$; by which, x, and consequently the Point B, may be determined: And, if $a=b$, it will be $x^3+3ax^2-2a^3=0$; which divided by $x+a$, makes $x^2+2ax-2a^2=0$, or, $x^2+2ax=2a^2$; and, by solving the Quadratic, $x=\overline{3a^2}|^{\frac{1}{2}}-a$.

Or thus. BECAUSE the Subtangent CT (*Art.* 46.) is $=\dfrac{\overline{a+x}\times\overline{b^2x-x^3}}{ab^2+x^3}$, and AC is $=b-x$; therefore AT ($=$CT$-$CA) $=$ $\dfrac{\overline{a+x}\times\overline{b^2x-x^3}}{ab^2+x^3}-b+x=$ a *Maximum* (*Art.* 60.) the *Fluxion* of which is

$$\frac{\overline{ab^2\dot{x}-3ax^2\dot{x}+2b^2x\dot{x}-4x^3\dot{x}}\times\overline{ab^2+x^3}-:3x^2\dot{x}\times\overline{ab^2x-ax^3+b^2x^2-x^4}}{\overline{ab^2+x^3}|^2 \text{ or } a^2b^4+2ab^2x^3+x^6}$$

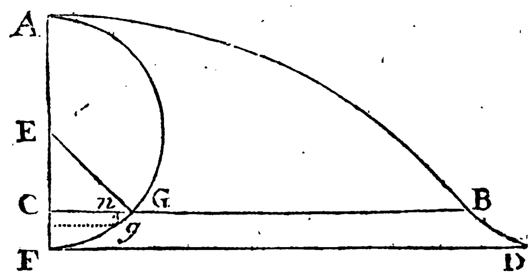
$+\dot{x}=0$: then multiply by $a^2b^4+2ab^2x^3+x^6$, and strike out the contradictory Terms, and it will be $2a^2b^4\dot{x}-4ab^2x^3\dot{x}-3a^2b^2x^2\dot{x}+2ab^4x\dot{x}-b^2x^4\dot{x}=0$; which divided by $-ab^2\dot{x}-b^2x\dot{x}$, gives $x^3+3ax^2-2ab^2=0$; as before.

EXAMPLE II.

62. *To find the Point of Inflection in the interior or inflected Cycloid, the Circumference of whose generating Circle is to its Base as 4 to 5.**

* *See how this Curve may be generated,* Appendix, *Prob.* 6.

PUT the generating Semicircle AGF$=a$, Base FD$=b$, Radius EA or EF$=r$, Abscifs AC$=x$, Ordinate CB $=y$, Arch AG $=z$,



and Sine GC$=s$; and suppose B to be the Point of Inflection sought. Now, by the Nature of the Curve, $a:b::z:$ (GB) $y-s$; and therefore (*Art.* 8.) $a:b::\dot{z}:\dot{y}-\dot{s}$; $\therefore a\dot{y}-a\dot{s}=b\dot{z}$, and $\dot{y}=\dot{s}+\frac{b\dot{z}}{a}$. Let the incremental Triangle Ggn, and the Radius EG be drawn; then will the Triangles ECG and Gng be alike: (for, Gg being considered as an indefinitely small right Line coinciding with a Tangent to the Point G, the $\angle$EGg will be a right Angle, and therefore $=$ $\angle$CEG $+$ $\angle$EGC; and, the $\angle$EGC being common, therefore $\angle$CEG $=$ $\angle n$Gg; and consequently, the $\angle$s GCE and Gng being right, the $\angle$EGC $=$ the $\angle$Ggn:) therefore,

as

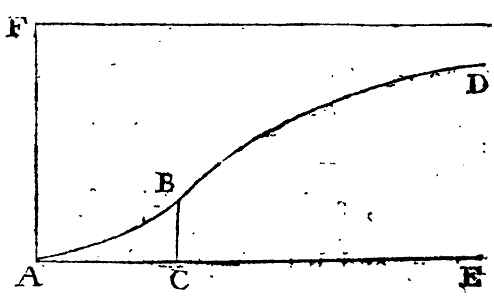
as $nG : ng :: EC : CG$, or (*Art.* 6. and 19.) $\dot{x}$ and $\dot{s}$ being Negative to each other, as $-\dot{s} : \dot{x} :: x-r : s$; $\because \dot{s} = \frac{\overline{r-x}\times\dot{x}}{s}$; but, by the Property of the Circle, $GC = \overline{AC\times CF}|^{\frac{1}{2}}$, i. e. $s = \overline{2rx-x^2}|^{\frac{1}{2}}$; which substituted for s, makes $\dot{s} = \frac{\overline{r-x}\times\dot{x}}{\overline{2rx-x^2}|^{\frac{1}{2}}}$. By 47 E. 1. $Gg = \overline{\overline{gn}|^2 + \overline{nG}|^2}|^{\frac{1}{2}}$, i. e. $\dot{z} = \overline{\dot{x}^2 + \dot{s}^2}|^{\frac{1}{2}} =$ (by substituting for $\dot{s}$, its above Value,) $\frac{r\dot{x}}{\overline{2rx-x^2}|^{\frac{1}{2}}}$. Hence, by writing for $\dot{s}$ and $\dot{z}$ these their Values, we have the above $\dot{y}$ $(= \dot{s} + \frac{b\dot{z}}{a}) = \frac{\overline{r-x}\times\dot{x}}{\overline{2rx-x^2}|^{\frac{1}{2}}} + \frac{br\dot{x}}{a\times\overline{2rx-x^2}|^{\frac{1}{2}}} = \frac{ar+br-ax}{a\times\overline{2rx-x^2}|^{\frac{1}{2}}}\dot{x}$; and this put into *Fluxions*, (supposing x to flow with an uniform Motion, when by *Art.* 57. $\ddot{x}$ and $\ddot{y}$, at the Point of Inflection, will be $=0$,) is $0 = \frac{brx-ar^2-br^2}{a\times\overline{2rx-x^2}|^{\frac{3}{2}}}\dot{x}^2$; consequently $brx - ar^2 - br^2 = 0$, and $brx = ar^2 + br^2$; therefore $x = \frac{a+b}{b}r$, and EC $(= x-r) = \frac{a}{b}r$; which gives the Point C in the Radius EF; from which if a perpendicular Ordinate be drawn, it

it will fall on the Point of Inflection sought. And this is an universal Theorem for all inflected Cycloids, when $\frac{a}{b}$ expresses the Ratio of the Circumference of the generating Circle to the Base of the Cycloid; which, in the present Case, being $\frac{4}{5}$, therefore $EC=\frac{4}{5}EF$. [*See* Appendix, *Prob.* II. Corol. 3.]

EXAMPLE III.

63. *To find the Point of Inflection* B, *in the Curve* ABD, *whose Equation (putting the Abscis* $AC=x$, *Ordinate* $CB=y$, *and the Perpendicular* $AF=a$,) *is* $ax^2=a^2y+x^2y$.

THE Equation of the Curve put into *Fluxions*, is $2ax\dot{x}=a^2\dot{y}+2x\dot{x}y+x^2\dot{y}$; there-



fore $\dot{y}=\frac{2ax\dot{x}-2x\dot{x}y}{a^2+x^2}$, i. e. (by substituting

ax^2

$\frac{ax^2}{a^2+x^2}$ for its Value y,) $\dot{y} = \frac{2a^3x\dot{x}}{\overline{a^2+x^2}|^2}$;
and the *Fluxion* of this again, making $\ddot{x}$ and $\ddot{y}=0$, is $0 =$

$$\frac{2a^3\dot{x}^2 \times \overline{a^2+x^2}|^2 - :\overline{4a^2x\dot{x}+4x^3\dot{x}} \times 2a^3x\dot{x}}{\overline{a^2+x^2}|^4};$$

therefore $2a^3\dot{x}^2 \times \overline{a^2+x^2}|^2 - .\overline{4a^2x\dot{x}+4x^3\dot{x}} \times 2a^3x\dot{x}=0$; and by dividing this by $2a^3\dot{x}^2$, we shall have $\overline{a^2+x^2}|^2 - 4a^2x^2 - 4x^4 = 0$; therefore $4x^4 + 4a^2x^2 = \overline{a^2+x^2}|^2$; and this Equation divided by x^2+a^2, makes $4x^2 = a^2+x^2$; by Tranſpoſition, $3x^2=a^2$; $\because x^2 = \frac{1}{3}a^2$, and $x=a\sqrt{\frac{1}{3}}$; and if this be ſubſtituted for x in the Equation of the Curve, we ſhall have y, or, the Ordinate at the Point of Inflection $= \frac{1}{4}a$.

IF it were required to find the Aſymptote of this Curve; we need only ſuppoſe the Abſciſs and Curve to be indefinitely extended: for then, becauſe x^2 will be indefinitely near to Equality with a^2+x^2, we ſhall have y (which by Equation of the Curve is $=a \times \frac{x^2}{a^2+x^2}$,) indefinitely near to Equality with the given right Line a; that is, y will then be $= a$, *Minus* a Quantity indefinitely ſmall. Wherefore, if from the Point F, a right

right Line be drawn parallel to the Axis AE, it will be the Aſymptote required.

64. *Note.* IN order to know whether an inflected Curve, at any Point aſſigned, be *Concave* or *Convex* towards its Axis; find the Value of $\ddot{y}$ at that Point: Then, (*Art.* 57.) if this Value of $\ddot{y}$ comes out *Poſitive*, the Curve will be *Convex* towards the Axis; and, if it comes out *Negative*, the Curve will be *Concave*.

Thus, in the laſt Example, if it were required to find whether the Curve be at firſt *Concave* or *Convex* towards the Axis: Suppoſing $a=10$; make $x=1$, or $x=$any Number leſs than $a\sqrt{\frac{1}{3}}$ or $10\sqrt{\frac{1}{3}}$; then, becauſe $\ddot{y}$ is always as $\overline{a^2+x^2}|^2-4a^2x^2-4x^4$, this Value of $\ddot{y}$ will come out *Affirmative*; and therefore the Curve at firſt is *Convex* towards the Axis: And when x is $=6$, or any Number greater than $a\sqrt{\frac{1}{3}}$ or $10\sqrt{\frac{1}{3}}$, this Expreſſion will come out *Negative*; and therefore, then, the Curve will be *Concave* towards the Axis.

CHAP.

CHAP. VI.

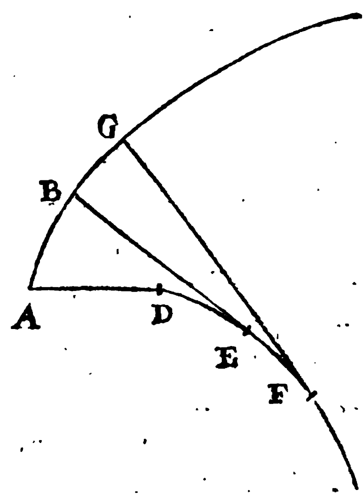
Of finding the Radius of Curvature.

65. AS the Curvature, or Convexity, of all Curves but Circles, varies in every Point; therefore, if Circles are drawn to coincide with a given Curve in any Number of Points; the Radii of theſe Circles will be different: And the finding of theſe Radii, is the Buſineſs of this Chapter.

66. AND, becauſe all Curves but Circles, are formed or generated, or may be conceived to be formed or generated, by the Evolution or winding off of ſome other Curves; therefore, the Centers of the Circles which coincide (or have equal Degrees of Curvature) with the different Points, or rather Increments, of the Curves thus formed or generated, will always be in the Curves to be unwound, which Curves are called the *Evolutes*; and the others formed or generated, or conceived to be formed or generated by their Evolution, are called the *Involutes*.

67. THUS, let DEF be any Curve, called an *Evolute*; round which, conceive a Thread to be wound and extended beyond the Curve, from D in a right Line to A; and

let this Thread be evolved, or wound off, from the Curve DF, ſo that it be continually ſtretched at its full Length as it leaves the Curve: Then will the Point A generate, or



deſcribe, the *Involute* Curve ABG; and the right Lines A D, B E, G F, will be the *Radii of Curvature* at the Points A, B, G, reſpectively.

COROLLARIES.

1. THE Radius of (Evolution or) Curvature BE, will always be equal to the Length of the Curve ED and right Line DA: And, conſequently, if the vertical Diſtance, or ſhorteſt

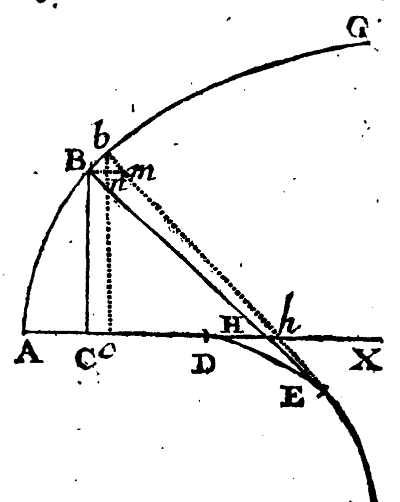
shortest Radius DA, vanish; that is, if the Radius at A be Nothing; then, the involute Curve will begin at D; and so, the Curve DE will be equal to the Radius of (Evolution or) Curvature at the Point B.

2. Because the Radius of a Circle is perpendicular to the Tangent, the Radius of Curvature at any Point B, is always perpendicular to a Tangent at that Point.

3. The same Radius BE, which is perpendicular to the Involute at the Point B, is also a Tangent to the Evolute at the Point E.

PROB.

To deduce a general Expression for BE, *the Radius of Evolution, or Curvature, for any*



 Point

Point B *of the Involute Curve* ABG, *whose Axis is* AX, *and Evolute* DE.

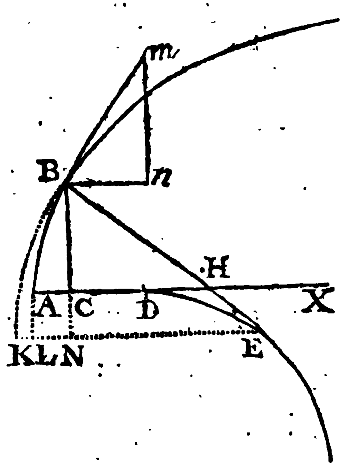
68. Put the Abſciſs AC $= x$, and Ordinate CB$=y$: Suppoſe bE indefinitely near to BE, bc indefinitely near and parallel to BC, and Bnm parallel to AX; i. e. let Cc or B$n=x'$, and $nb=y'$: Then, Bb being conſidered as a little right Line coinciding with a Tangent at the Point B, the Triangles Bnb and BCH will be ſimilar; (for, $\angle$EB$b=\angle$CBn, and therefore, the $\angle$EBn being common, the $\angle$s nBb and CBH are equal; $\because$ $\angle$B$bn=\angle$CHB; *ergo*, &c.) and, the $\angle$Bbm being right, the Triangles mnb and bnB will be ſimilar alſo; Wherefore, Bn : nb :: BC : CH; that is, $x' : y' :: y : \frac{yy'}{x'} =$ CH; and therefore (by 47 E. 1.) BH $= \overline{\overline{BC}|^2 + \overline{CH}|^2}|^{\frac{1}{2}} = \overline{y^2 + \frac{y^2y'^2}{x'^2}}\Big|^{\frac{1}{2}} = \frac{y}{x'} \times \overline{x'^2 + y'^2}|^{\frac{1}{2}}$; and AH$=x+\frac{yy'}{x'}$. Again, B$n$: nb :: bn : nm; i.e. $x' : y' :: y' : \frac{y'^2}{x'} = nm$; $\because$ B$m = x' + \frac{y'^2}{x'}$. Now, becauſe the Direction of the Curve ABG approaches continually

continually nearer to a Parallelifm with the Axis AX; therefore, if we fuppofe the Abfcifs ($AC = x$) to flow with an equable or uniform Motion, i. e. fuppofing x' or $\dot{x}$ to be invariable, or always of the fame Value; then, the *Increment* of the Ordinate ($CB = y$) or the *Velocity* or *Fluxion* with which it flows, muft continually decreafe; that is, the *Second Increment* or *Second Fluxion* of y will be Negative: Therefore, then, Hb, the *Increment* of AH, *viz.* the *Increment* of $x + \frac{yy'}{x'}$, will be $= x' + \frac{y'^2 - yy''}{x'}$. Now, the Triangles EBm and EHb are fimilar, as is evident; therefore $Bm - Hb : Bm :: (BE - HE =)$ $BH : BE$; i. e. $\frac{yy''}{x'} : x' + \frac{y'^2}{x'} :: \frac{y}{x'} \times \overline{x'^2 + y'^2}|^{\frac{1}{2}} :$ $\frac{\overline{x'^2 + y'^2}|^{\frac{3}{2}}}{x'y''} = BE$; or, fubftituting the *Fluxions* for the *Increments* (*Article* 6.) $BE =$ $\frac{\overline{\dot{x}^2 + \dot{y}^2}|^{\frac{3}{2}}}{\dot{x}\ddot{y}}$.

69. *Or* (without the Aid of *Increments*) *thus*. With the Radius E B, defcribe the circular Arch B K, which will therefore have the fame Degree of Curvature as the

Involute

Involute Curve A B at the Point B. Draw the Radius EK parallel to the Axis AX; and produce the Ordinate B C to N, to which draw AL parallel. Put the Abscis AC $= x$, Ordinate CB $= y$, Radius E B or EK $= r$, KL $= a$, and LA $= b$; then NE $= r - a - x$.



Now, if we suppose the Abscis (x) to increase uniformly, and Bm to be a Tangent at the Point B; then, if we draw mn parallel to BC, and Bn parallel to AX, by *Art.* 37. Bn, nm, and mB, will be as the *Fluxions* of the Abscis, Ordinate, and Curve, respectively; that is, B n will be as $\dot{x}$, $n\,m$ as $\dot{y}$, and (because by 47 E. 1. B$m = \overline{Bn^2 + nm^2}|^{\frac{1}{2}}$,) B m as $\overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}}$. Now, the Triangles B n m and

BNE

BNE are similar; therefore B*n* : *nm* :: BN : NE, i. e. $\dot{x} : \dot{y} :: y+b : r-a-x$; ∴ $r\dot{x}-a\dot{x}-x\dot{x}=y\dot{y}+b\dot{y}$; the *Fluxion* of which Equation (supposing $\dot{x}$ invariable, and therefore, the Direction of the Curve AB continually approaching towards a Parallelism with its Axis, the *Fluxion* of $\dot{y}$ as Negative,) is $-\dot{x}^2=\dot{y}^2-y\ddot{y}-b\ddot{y}$; ∴ $\dot{x}^2+\dot{y}^2=\overline{b+y}\times\ddot{y}$. Again, by Sim. △s, N B : B E :: *n* B : B*m*, i. e. $b+y : r :: \dot{x} : \overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}$; ∴ $b+y=\frac{r\dot{x}}{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}}$; which substituted for $b+y$, makes the above $\dot{x}^2+\dot{y}^2=\frac{r\dot{x}\ddot{y}}{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}}$; therefore $\overline{\dot{x}^2+\dot{y}^2}\times\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}=r\dot{x}\ddot{y}$, i. e. $\overline{\dot{x}^2+\dot{y}^2}|^{\frac{3}{2}}=r\dot{x}\ddot{y}$; ∴ $\frac{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{3}{2}}}{\dot{x}\ddot{y}}=r=$ BE; as before.

70. *Note.* When the Abscifs, x, flows with an uniform Motion, it follows from *Art.* 57. that the Ordinate, y, flows with a *retarded* Motion when it increases and the Curve is Concave, or when it decreases and the Curve is Convex towards the Axis; and with an *accelerated* Motion when it decreases and the Curve is Concave, or when it increases and the Curve is Convex towards the Axis. Now, when y increases with an accelerated

celerated Motion, or its *Second Fluxion* is Affirmative, the general Expression for the Radius of Curvature will be $=\frac{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$.

71. HENCE, because we may substitute 1 for any invariable *Fluxion*, if we put $\dot{x}=1$, the general Expression for the Radius of Curvature will be $=\frac{\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{\ddot{y}}$ when y increases with a retarded Motion, or its *Second Fluxion* is Negative: And $=\frac{\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{-\ddot{y}}$ when y increases with an accelerated Motion, or its *Second Fluxion* is Affirmitive. The former takes Place when the Curve is *Concave*, and the latter when it is *Convex* towards the Axis.——Wherefore, if we put the Equation of the given Curve expressing the Relation between the Abscifs x and Ordinate y into *Fluxions*, making $\dot{x}=1$; or from the Nature of the Curve, find the Value of $\dot{x}=1$, in Terms of x, y, and $\dot{y}$; and then put this fluxional Equation into *Fluxions* again, still substituting 1 for $\dot{x}$, and making the *Fluxion* of $\dot{y}$ *Negative* when the Curve is *Concave*, and *Affirmative* when it is *Convex* towards its Axis; from thence, the Values

lues of the *Second* and *Square* of the *First* Fluxions of y may be had or determined: Which therefore being substituted for them in one of these general Expressions, *viz.* in the *First* when the *Fluxion* of $\dot{y}$ is Negative; and in the *Second* when it is Affirmative: We shall then have a definitive Expression for BE, freed from *Fluxions*, or, the Radius of Curvature required.

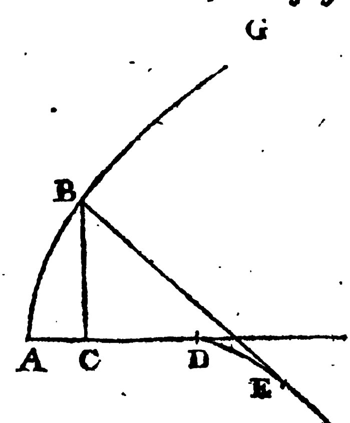
72. *N. B.* The shortest Radius, or vertical Distance, AD, may be obtained, by writing for $\dot{x}$ and $\dot{y}$ their Values, in the general Expression for the Subnormal CH, which was found in *Article* 68. ($=\frac{yy'}{x'}$, i.e. by substituting $\dot{x}$ for x', and $\dot{y}$ for y',)$=\frac{y\dot{y}}{\dot{x}}$; or, making $\dot{x}=1$, by substituting the Value of $\dot{y}$ in $y\dot{y}$, i. e. by multiplying the Value of $\dot{y}$ by y; and then making x and y vanish in the definitive Expression, which will then be found.—— For, the Expression for the Subnormal CH being the same, at whatever Point of the Curve B is taken; therefore,

fore, if it be taken at A, where x and y vanish, or become = o, the Point C must of consequence coincide with the Point or Vertex A, and the Points E and H, with D: *ergo, &c.*

73. *Note.* THE substituting *Unity*, or 1, rather than any other Number, for an *invariable Fluxion*, has no Manner of Effect on the Working, otherwise than by making the Operation much less laborious; and, in reality, it is no more than making *Unity* the Standard of the other *Fluxions*, or reducing the other *Fluxions* to a Comparison with 1.

EXAMPLE I.

74. *To find a definitive Expression for the Radius of Curvature* BE; *as also for* AD, *the*



Distance

Distance of the Vertices of the Evolute and Involute Curves: Supposing the Involute Curve ABG *to be the conic or common apollonian Parabola.*

THE Equation of the common Parabola is $ax = y^2$, (See *Art.* 41.) the *Fluxion* of which is $a\dot{x} = 2y\dot{y}$, or, making $\dot{x} = 1$, it is $a = 2y\dot{y}$; therefore $\dot{y} = \frac{a}{2y} = \frac{a}{2 \times \overline{ax}|^{\frac{1}{2}}}$, for y is $= \overline{ax}|^{\frac{1}{2}}$ by the Equation of the Curve: Now, the *Fluxion* of this Equation again, (the Direction of the Curve approaching continually towards a Parallelism with the Axis, and therefore the *Fluxion* of $\dot{y}$ being Negative, *Art.* 57.) is $-\ddot{y} = \frac{-a^2}{4 \times \overline{ax}|^{\frac{3}{2}}}$: Whence, $\dot{y}^2 = \frac{a^2}{4 \times ax} = \frac{a}{4x}$, and $\ddot{y} = \frac{a^2}{4 \times \overline{ax}|^{\frac{3}{2}}}$. Now, if for $\dot{y}^2$ and $\ddot{y}$ we substitute these their Values, we shall have $\frac{\overline{1 + \dot{y}^2}|^{\frac{3}{2}}}{\ddot{y}}$, the general Expression for the Radius of Curvature BE

$$(\textit{Art. } 71.) = \frac{\overline{1 + \frac{a}{4x}}|^{\frac{3}{2}} \times 4 . \overline{ax}|^{\frac{3}{2}}}{a^2} = \frac{\overline{4ax + a^2}|^{\frac{3}{2}}}{2a^2};$$

which is the definitive Expression required.

 And,

And, by substituting for $\dot{y}$ its Value $\frac{a}{2y}$, in $y\dot{y}$ (*Art.* 72.) we have $\frac{a}{2}$ or $\frac{1}{2}a =$ AD the vertical Distance: Which same Truth may be inferred from the Expression for the Radius BE; for, when the said Radius becomes the vertical Distance, that is, when the Point B coincides with A, x vanishes; and therefore, by striking $4ax$ out of the said Expression, we have $\frac{\overline{a^2}|^{\frac{3}{2}}}{2a^2} = \frac{1}{2}a$; as before.

Example II.

75. *Let* $y^m = x$ *express the Nature of all Parabolas universally,* (*See* Art. 42.) *To find the Radius of Curvature (or of Evolution,) and the vertical Distance; as also, the Consequences on these three Suppositions respectively, viz.* $m = 2$, m *more than* 2, m *less than* 2.

The *Fluxion* of this Equation (making $\dot{x} = 1$,) is $my^{m-1}\dot{y} = 1$; and the *Fluxion* of this again (supposing the *Fluxion* of $\dot{y}$ Negative,) is $\overline{m-1} \times my^{m-2}\dot{y}^2 - my^{m-1}\ddot{y} = 0$. Hence

$\ddot{y} =$

$\dot{y} = \frac{1}{my^{m-1}}$, and $\dot{y}^2 = \frac{1}{m^2 y^{2m-2}}$, and $\ddot{y} = \frac{\overline{m-1} \times m y^{m-2}}{m y^{m-1}} \dot{y}^2 =$ (by dividing both the Numerator and Denominator by my^{m-2};) $\frac{m-1}{y} \dot{y}^2 =$ (by writing for $\dot{y}^2$ its equal,) $\frac{m-1}{m^2 y^{2m-1}}$. Now, by substituting for $\dot{y}^2$ and $\ddot{y}$ these their Values, we shall have the general Expression for the Radius of Curvature, *viz.*

$$\frac{\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{\ddot{y}}, (\textit{Art.} 71.) = \frac{\overline{1 + \frac{1}{m^2 y^{2m-2}}}|^{\frac{3}{2}} \times m^2 y^{2m-1}}{m-1} = \frac{\overline{m^2 y^{2m-2} + 1}|^{\frac{3}{2}}}{\overline{m-1} \times m y^{m-2}}$$

$=$ the Radius of Curvature required. And by substituting $\frac{1}{my^{m-1}}$ for $\dot{y}$, in $y\dot{y}$ (*Art.* 72.) we have the Subnormal $= \frac{y}{my^{m-1}} = \frac{y^2}{my^m} = \frac{y^{2-m}}{m}$; that is, when $m = 2$, the Subnormal is $= \frac{y^0}{2} = \frac{1}{2} =$ the vertical Distance; and

when

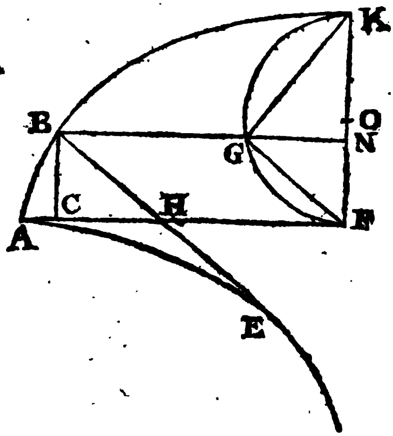
when m is *more* than 2, then $2-m$ will be a negative Index, and consequently the proper Place of y will be in the Denominator, and therefore, if y be made $=0$, the Expression will be infinite, and equal to the vertical Distance: And lastly, when m is *less* than 2, then $2-m$ is a positive Index, and y's proper Place is in the Numerator, as in the Expression; and therefore, when the Subnormal is equal to the vertical Distance, or $y=0$, the said vertical Distance will be Nothing.

Example III.

76. *To find the Radius of Curvature in that most beautiful mechanical Curve the Cycloid* *; *as also, the Distance of its Vertex from that of the Evolute.*

Put OF or OK $=a$, AC $=x$, CB $=y$, Sine NG $=s$, and Arch FG $=z$. Now, by the Property of Circles, NG $=\overline{KN\times NF}|^{\frac{1}{2}}$, i. e. $s=\overline{2ay-y^2}|^{\frac{1}{2}}$; the *Fluxion* of which Equation

* See how this Curve is generated, *Article* 48. *Note*.



Equation is $\dot{s} = \frac{a\dot{y} - y\dot{y}}{\overline{2ay - y^2}|^{\frac{1}{2}}}$. And, by the Nature of the Cycloid, (See *Art.* 48.) Arch KG = GB; and therefore, Arch FG = GN + AC, or AC = Arch FG — GN; i. e. $x = z - s =$ (by substituting for s its above Value,) $z - \overline{2ay - y^2}|^{\frac{1}{2}}$; and the *Fluxion* of this Equation (making $\dot{x} = 1$,) is $1 = \dot{z} + \frac{y\dot{y} - a\dot{y}}{\overline{2ay - y^2}|^{\frac{1}{2}}}$. But (*Article* 69.) $\dot{z} = \overline{\dot{s}^2 + \dot{y}^2}|^{\frac{1}{2}} =$ (by writing for $\dot{s}$ its above Value,) $\overline{\left(\frac{a\dot{y} - y\dot{y}}{2ay - y^2}\right)^2 + \dot{y}^2}|^{\frac{1}{2}} = \frac{a\dot{y}}{\overline{2ay - y^2}|^{\frac{1}{2}}}$; which substituted for $\dot{z}$, makes the above Equation

$1 =$

$1 = \frac{a\dot{y}}{\overline{2ay - y^2}|^{\frac{1}{2}}} + \frac{y\dot{y} - a\dot{y}}{\overline{2ay - y^2}|^{\frac{1}{2}}}$; that is, $1 = \frac{y\dot{y}}{\overline{2ay - y^2}|^{\frac{1}{2}}}$; therefore $\dot{y} = \frac{\overline{2ay - y^2}|^{\frac{1}{2}}}{y}$; and the *Fluxion* of this again (the *Fluxion* of $\dot{y}$ being Negative,) is $- \ddot{y} =$

$$\frac{\frac{ay\dot{y} - y^2\dot{y}}{\overline{2ay - y^2}|^{\frac{1}{2}}} - \dot{y} \times \overline{2ay - y^2}|^{\frac{1}{2}}}{y^2} = \frac{-a\dot{y}}{y \times \overline{2ay - y^2}|^{\frac{1}{2}}} =$$

(by writing for $\dot{y}$ its equal,) $- \frac{a}{y^2}$; that is, $\ddot{y} = \frac{a}{y^2}$. Now, by substituting $\frac{2ay - y^2}{y^2}$ for $\dot{y}^2$, and $\frac{a}{y^2}$ for $\ddot{y}$, we shall have, by

Art. 71. $\frac{\overline{1 + \dot{y}^2}|^{\frac{3}{2}}}{\ddot{y}} = \frac{\overline{1 + \frac{2ay - y^2}{y^2}}|^{\frac{3}{2}} \times y^2}{a} =$

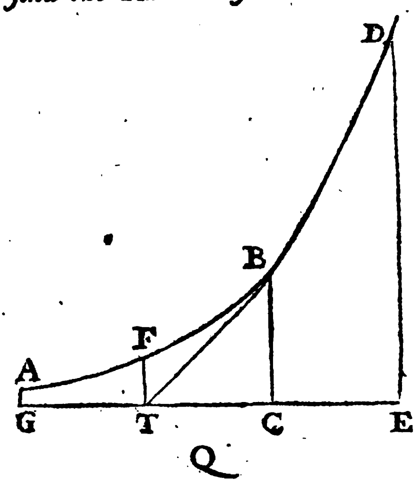
$\frac{\overline{2ay}|^{\frac{3}{2}}}{ay} =$ BE, the Radius of Curvature required. Whence, by Analogy, BE : $\overline{2ay}|^{\frac{1}{2}}$: : $2ay : ay$, that is, BE : $\overline{2ay}|^{\frac{1}{2}}$: : 2 : 1; but $\overline{2ay}|^{\frac{1}{2}}$ = Chord F G; (for, by 47 E. 1, $\overline{GN}|^2 + \overline{NF}|^2 = \overline{FG}|^2$, and, by the Property of Circles, $\overline{GN}|^2 = KN \times NF$; therefore,

$\overline{FG}|^2$

$\overline{FG}|^2 = KN \times NF + \overline{NF}|^2 = \overline{2a-y} \times y + y^2 = 2ay$, and consequently $FG = \overline{2ay}|^{\frac{1}{2}}$;) therefore BE=2GF: And, as a Tangent to the Point B is parallel to the Chord KG, by *Art.* 48. and the Angle KGF is right, by the Property of Circles; therefore, by *Art.* 67. *Cor.* 2. the Radius of Curvature BE will be parallel to the Chord GF. —— By *Article* 72. if we substitute $\frac{\overline{2ay-y^2}|^{\frac{1}{2}}}{y}$ for its equal $\dot{y}$, in $y\dot{y}$; we shall have the Subnormal CH$=\overline{2ay-y^2}|^{\frac{1}{2}}$; which, when y vanishes, becomes$=0$, and equal to the vertical Distance: So that the Vertices of the evolute and involute Curves coincide.

EXAMPLE IV.

77. *To find the Radius of Curvature at any*



Point

Point B *of the Curve* AD; *whose Nature is such, that the Triangle* CBT, *made of the Ordinate, Tangent, and Subtangent, is always proportional to the Ordinate* CB; *or, whose Subtangent* CT *is always the same.* (*See* Art. 50.)

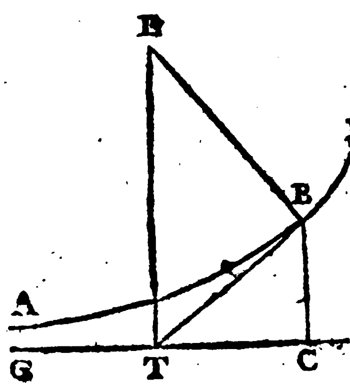
PUT the given Subtangent CT $=a$, GC $=x$, CB $=y$; then (by *Art.* 38.) $\frac{\dot{x}y}{\dot{y}}=a$, that is, (if $\dot{x}$ be made $=1$,) $\frac{y}{\dot{y}}=a$; $\because \dot{y}=\frac{y}{a}$; therefore $\dot{y}^2=\frac{y^2}{a^2}$, and (because here y flows with an accelerated Motion, or, its *Second Fluxion* is Affirmative,) $\ddot{y}=\frac{\dot{y}}{a}$, i. e. (by substituting $\frac{y}{a}$ for $\dot{y}$ its equal,) $\ddot{y}=\frac{y}{a^2}$. Now, by writing for $\dot{y}^2$ and $\ddot{y}$ these their Values, in $\frac{\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{-\ddot{y}}$, (*Article* 71.) we shall have $\frac{\overline{1+\frac{y^2}{a^2}}|^{\frac{3}{2}}\times a^2}{-y}=\frac{\overline{a^2+y^2}|^{\frac{3}{2}}}{-ay}$ $=$ the Radius of Curvature sought: Where the negative Sign, only shews, that the Evolute and Radius of Curvature lie on the other

other Side of the Curve with regard to x and y.

78. *Note.* The above Curve is the *logarithmic Curve*, whose Asymptote is E G: which is so called, because when the said Asymptote is divided into any Number of equal Parts, as in the Points G, T, C, E, the Ordinates to these Points will be in geometrical Progression; that is, GT, GC, &c. will be the *Logarithms* of the Ordinates TF, CB, &c.

EXAMPLE V.

79. *To find the Radius of Curvature at any Point* B *of the Curve* AD; *whose Nature*



is such, that the Tangent BT *is every where equal to the same given Line* $= a$.

PUT GC $= x$, CB $= y$; then, by 47 E. 1. $\overline{TB|^2 - BC|^2}|^{\frac{1}{2}} =$ C T, i. e. $\overline{a^2 - y^2}|^{\frac{1}{2}} =$ C T $=$ (by *Art.* 38.) $\frac{\dot{x}y}{\dot{y}}$, or (making $\dot{x} = 1$,) $\overline{a^2 - y^2}|^{\frac{1}{2}} = \frac{y}{\dot{y}}$; $\because \dot{y} = \frac{y}{\overline{a^2 - y^2}|^{\frac{1}{2}}}$; therefore, $\dot{y}^2 = \frac{y^2}{a^2 - y^2}$, and (because the *Fluxion* of $\dot{y}$ is Affirmative,) $\ddot{y} =$

$$\frac{\dot{y} \times \overline{a^2 - y^2}|^{\frac{1}{2}} + \frac{y^2 \dot{y}}{\overline{a^2 - y^2}|^{\frac{1}{2}}}}{a^2 - y^2},$$

i. e. (by substituting for $\dot{y}$ its equal,) $\ddot{y} = \frac{y + \frac{y^3}{a^2 - y^2}}{a^2 - y^2} = \frac{a^2 y}{\overline{a^2 - y^2}|^2}$. Now, by writing for $\dot{y}^2$ and $\ddot{y}$ these their Values, in $\frac{\overline{1 + \dot{y}^2}|^{\frac{3}{2}}}{-\ddot{y}}$, (See *Art.* 71.) we shall have

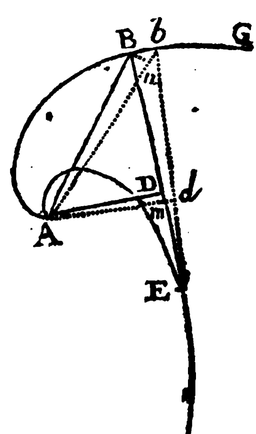
$$\frac{\overline{1 + \frac{y^2}{a^2 - y^2}}|^{\frac{3}{2}} \times \overline{a^2 - y^2}|^2}{-a^2 y},$$

that is, $\frac{\overline{a^2}|^{\frac{3}{2}} \times \overline{a^2 - y^2}|^{\frac{1}{2}}}{-a^2 y}$, or, $-\frac{a}{y} \times \overline{a^2 - y^2}|^{\frac{1}{2}} =$ the Radius of Curvature required: Where the negative Sign shews its Position. Hence the following Construction; *viz.* On the Extremity of the Subtangent, T, erect the Per-

Perpendicular TE; and draw the right Line BE perpendicular to the Tangent TB: then will BE be the Radius of Curvature at the Point B; or, the Point E will be in the Evolute Curve: For, the Triangles CBT and BTE will be similar; and consequently, BC : CT : : TB : BE, i. e. $y : \overline{a^2 - y^2}|^{\frac{1}{2}} : : a : \frac{a}{y} \times \overline{a^2 - y^2}|^{\frac{1}{2}} = BE$.

THE general Expression for the Radius of Evolution, or Curvature, found *Art.* 71, being only for Curves referred to an Axis; we shall now therefore deduce one for *Spirals*, or those Curves whose Ordinates are referred to a fixt or central Point.

80. LET ABG be the Curve; A, the



central Point, or that from which all the

Ordinates

Ordinates issue; and BE, the Radius of Curvature at the Point B, i. e. let the Point B be supposed in the Evolute Curve: Conceive Ab and Eb indefinitely near to AB and EB, i. e. let the Points B and b be supposed indefinitely near to each other; and let AD and Ad be perpendicular to EB and Eb respectively: then will the Points D and m be indefinitely near to a Coincidence; and therefore (*Art.* 6.) B m and A m may be taken as equal to BD and AD. Now, if with the Ordinate AB, as a Radius, the little circular Arch B n be described, and considered as a little right Line perpendicular to Ab; and the *Increment* Bb be considered as coinciding with a Tangent to the Point B; then, the little right angled Triangle bnB will be similar to the right angled Triangle BDA; (for $\angle ABn = \angle EBb$; and therefore, $\angle EBn$ being common, the $\angle ABD = \angle nBb$; and consequently, the Angles at D and n being right, $\angle BAD = \angle Bbn$;) therefore, $b\,B : B\,n :: A\,B : B\,D$; that is, (if we put the Ordinate $A\,B = y$, $Bn = x'$, and $nb = y'$, when by 47 E. 1. $B\,b$ will be $= \overline{x'^2 + y'^2}\big|^{\frac{1}{2}}$,) $\overline{x'^2 + y'^2}\big|^{\frac{1}{2}} : x' :: y : \dfrac{x'y}{\overline{x'^2 + y'^2}\big|^{\frac{1}{2}}} =$ BD or B m; or (substituting $\dot{x}$

for

for x', and $\dot{y}$ for y',) $\frac{\dot{x}y}{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}}$ = BD or B*m*.

Again, B*b* : *bn* :: BA : AD; that is,

$\overline{x'^2+y'^2}|^{\frac{1}{2}} : y' :: y : \frac{y'y}{\overline{x'^2+y'^2}|^{\frac{1}{2}}}$ = AD or A*m*;

or, $\frac{y\dot{y}}{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}}$ = AD or A*m*; the *Fluxion* of which is expressed by the *Increment md*; that is, (supposing $\dot{x}$ to be invariable,)

$$\frac{\overline{\dot{y}^2+y\ddot{y}}\times\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}-\frac{\dot{y}\ddot{y}\times y\dot{y}}{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}}}{\dot{x}^2+\dot{y}^2}$$

$\frac{\dot{x}^2\dot{y}^2+\dot{y}^4+y\dot{x}^2\ddot{y}}{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{3}{2}}}$ = *md*. But, by the similar Triangles EB*b* and E*md*, we have B*b* — *md* : B*b* :: (BE — *m*E, or) *m*B : BE; that is,

$$(\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}-\frac{\dot{x}^2\dot{y}^2+\dot{y}^4+y\dot{x}^2\ddot{y}}{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{3}{2}}})$$

$$\frac{\dot{x}^4+\dot{x}^2\dot{y}^2-y\dot{x}^2\ddot{y}}{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{3}{2}}} : \overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}} :: \frac{\dot{x}y}{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}} :$$

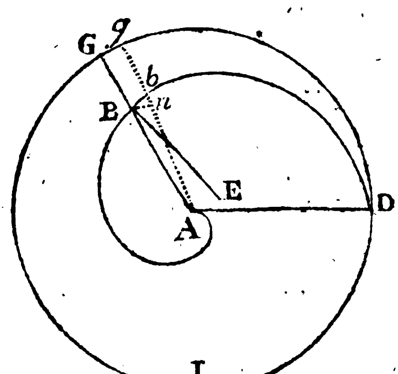
$\frac{y\times\overline{\dot{x}^2+\dot{y}^2}|^{\frac{3}{2}}}{\dot{x}^3+\dot{x}\dot{y}^2-y\dot{x}\ddot{y}}$ = BE; which is a general Expression for the Radius of Evolution, or Curvature, of all Curves referred to a fixt or central Point, when $\dot{x}$ is invariable. Wherefore,

81. If $\dot{x}$ be made $=1$, the general Expression for BE, the Radius of Curvature, will be $= \frac{y \times \overline{1+\dot{y}^2}|^{\frac{3}{2}}}{1+\dot{y}^2-y\ddot{y}}$. And, if we put the Equation of the given Spiral into *Fluxions*, (making $\dot{x}=1$,) and put this fluxional Equation into *Fluxions* again; and from thence, or from the Nature of the Curve, find the Values of $\dot{y}^2$ and $\ddot{y}$: then, if for $\dot{y}^2$ and $\ddot{y}$, we substitute these their Values, in this general Expression, we shall have the Radius of Curvature required: As in the following Examples.

EXAMPLE I.

82. *To find the Radius of Curvature at any Point* B *of the Spiral of* Archimedes, ABD.*

Put



* See how this Curve is generated, *Art.* 54. *Note.*

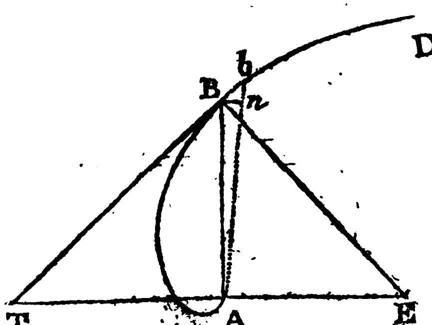
PUT the Circumference of the generating Circle DIGD $= a$, and its Radius AD or AG $= b$; Ordinate AB $= y$, Arch DIG $= z$. Let Ag be suppoſed indefinitely near to AG, i. e. let the $\angle$GAg be ſuppoſed indefinitely ſmall; and with the Ordinate AB, as a Radius, deſcribe the little circular Arch Bn, which put $= x'$; alſo, put G$g = z'$. Now, by the Nature of the Curve, $a : b :: z : y$, or, $z = \frac{ay}{b}$; the *Fluxion* of which Equation is $\dot{z} = \frac{a\dot{y}}{b}$: But, by the ſimilar Sectors ABn and AGg, we have $y : x' :: b : z' = \frac{bx'}{y}$, or (*Article* 6.) $\dot{z} = \frac{b\dot{x}}{y}$. Hence, $\frac{a\dot{y}}{b} = \frac{b\dot{x}}{y}$; that is (making $\dot{x} = 1$,) $\frac{a\dot{y}}{b} = \frac{b}{y}$; which Equation reduced, gives $\dot{y} = \frac{b^2}{ay}$; therefore $\dot{y}^2 = \frac{b^4}{a^2y^2}$, and $\ddot{y} = \frac{-ab^2\dot{y}}{a^2y^2} =$ (by writing for $\dot{y}$ it Value $\frac{b^2}{ay}$,) $\frac{-b^4}{a^2y^3}$: And, if we ſubſtitute for $\dot{y}^2$ and $\ddot{y}$ theſe their Values, we ſhall have $\frac{y \times \overline{1+\dot{y}^2}|^{\frac{3}{2}}}{1+\dot{y}^2 - y\ddot{y}}$ (*Article* 81.) $=$

$$\frac{y \times \overline{1+\frac{b^4}{a^2y^2}}\Big|^{\frac{3}{2}}}{1+\frac{b^4}{a^2y^2}+\frac{b^4}{a^2y^2}} = \frac{\overline{a^2y^2+b^4}\Big|^{\frac{3}{2}}}{a^3y^2+2ab^4} = BE,$$ the Radius of Curvature sought.

EXAMPLE II.

83. *To find the Radius of Curvature at any Point* B *of the logarithmic Spiral* ABD; *whose Equation (putting the Ordinate* AB=y, *Curve* AB=z, *and* a *and* b *for two given Quantities,) is* az=by. *(See* Article 55.)

THE *Fluxion* of the Equation of the Curve is $a\dot{z}=b\dot{y}$; therefore $\dot{z}=\frac{b\dot{y}}{a}$. Let the An-



gle BA*b* be supposed indefinitely small; and with

with the Ordinate AB, as a Radius, let the little circular Arch Bn be described; which being considered as a little right Line perpendicular to Ab; and Bb as a little right Line coinciding with a Tangent to the Point B; we shall have, by 47 E. 1. $bB = \overline{\overline{Bn}|^2 + \overline{nb}|^2}|^{\frac{1}{2}}$, i. e. (putting $Bn = x'$, $nb = y'$, and $Bb = z'$,) $z' = \overline{x'^2 + y'^2}|^{\frac{1}{2}}$, or, by substituting the *Fluxion* for the *Increment*, $\dot{z} = \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}}$. Hence, $\frac{b\dot{y}}{a} = \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}}$, i. e. (if we put $\dot{x} = 1$,) $\frac{b\dot{y}}{a} = \overline{1 + \dot{y}^2}|^{\frac{1}{2}}$; which Equation squared is $\frac{b^2\dot{y}^2}{a^2} = 1 + \dot{y}^2$; and this produces $\dot{y}^2 = \frac{a^2}{b^2 - a^2}$; therefore, by Evolution, $\dot{y} = \frac{a}{\overline{b^2 - a^2}|^{\frac{1}{2}}}$; which being an invariable Quantity, therefore $\ddot{y} = 0$. Now, by writing for $\dot{y}^2$ and $\ddot{y}$ these their Values, the general Expression for the Radius of Curvature, *viz.* $\frac{y \times \overline{1 + \dot{y}^2}|^{\frac{3}{2}}}{1 + \dot{y}^2 - y\ddot{y}}$, (*Art.* 81.) will become $=$

$$\frac{y \times \overline{1+\frac{a^2}{b^2-a^2}}\big|^{\frac{3}{2}}}{1+\frac{a^2}{b^2-a^2}-0} = y\sqrt{\frac{b^2}{b^2-a^2}} = y \times$$

$\frac{b}{\sqrt{b^2-a^2}}$ = BE, the Radius of Curvature required. Hence, and from *Article* 55. we have the following Conſtruction: *viz.* Draw the right Line EAT perpendicular to the Ordinate A B; and make A T to T B as $\sqrt{b^2-a^2}$ is to b; then, drawing BE perpendicular to B T, the Point E will be in the evolute Curve, or B E will be the Radius of Curvature at the Point B: For then, by *Article* 55. BT will be a Tangent, and the Subtangent $AT = y \times \frac{\sqrt{b^2-a^2}}{a}$; and (*Article* 67. *Cor.* 2.) the Tangent and Radius of Curvature are always perpendicular to each other; therefore, becauſe the Triangles TAB and BAE are ſimilar, TA : AB : : BA : AE, i. e. $y \times \frac{\sqrt{b^2-a^2}}{a} : y :: y : \frac{ay}{\sqrt{b^2-a^2}} = AE$; but, by 47 E. 1. $BE = \overline{EA}|^2 + \overline{AB}|^2\big|^{\frac{1}{2}}$,

i. e.

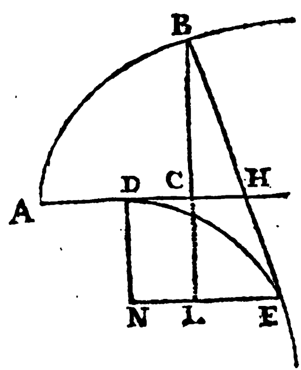
i. e. $BE = \overline{\frac{a^2y^2}{b^2-a^2}+y^2}\Big|^{\frac{1}{2}} = \overline{\frac{b^2y^2}{b^2-a^2}}\Big|^{\frac{1}{2}} = y \times \frac{b}{\sqrt{b^2-a^2}}$.

CHAP. VII.

Of finding the Nature of the Evolute *of a given* Involute Curve.

AS it is absolutely necessary for the Learner to be well acquainted with the foregoing Chapter before he enters upon this; we shall not here define the meaning of *evolute* and *involute* Curves, it being sufficiently explained therein.

84. LET BE be the Radius of Evolution at any Point B of the involute Curve AB,



whose

whose Abscis is AC$=x$, and Ordinate CB $=y$. Parallel to HA draw EN; produce BC to L; and, equal and parallel to CL, draw DN from the Vertex of the Evolute DE. Then will the Triangles BHC and BEL be similar; and therefore, BH : HC : : BE : EL, i. e. (by *Article* 68. writing the *Fluxion* for the *Increment*,) $\frac{y}{\dot{x}} \times \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}} : \frac{y\dot{y}}{\dot{x}} : : \frac{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{3}{2}}}{\dot{x}\ddot{y}} : \frac{\dot{y} \times \overline{\dot{x}^2+\dot{y}^2}}{\dot{x}\ddot{y}} =$ EL: Again, HC : CB : : EL : LB, i. e. $\frac{y\dot{y}}{\dot{x}} : y : : \frac{\dot{y} \times \overline{\dot{x}^2+\dot{y}^2}}{\dot{x}\ddot{y}} : \frac{\dot{x}^2 + \dot{y}^2}{\ddot{y}} =$ LB. Now, these are general Expressions for EL and LB, when $\dot{x}$ is considered as invariable, and the *Fluxion* of $\dot{y}$ as Negative. Hence therefore,

85. If $\dot{x} = 1$, and the *Fluxion* of $\dot{y}$ be Negative; the general Expression for BL will be $= \frac{1+\dot{y}^2}{\ddot{y}}$; and this multiplied by $\dot{y}$, is $\dot{y} \times \frac{1+\dot{y}^2}{\ddot{y}} =$ the general Expression for LE. Now, by Help of the Equation of the given involute Curve, exterminate $\dot{y}$, $\dot{y}^2$, and $\ddot{y}$, out of these Expressions, as in the preceding

Chapter;

Chapter; and, by *Article* 72. find the vertical Distane AD: Then, if we put the Absciss of the Evolute DN$=u$, and its Ordinate NE$=v$; by Help of these two Equations, $u=$BL$-$BC, and $v=$AC$-$AD$+$LE, we may get the Nature of the evolute Curve DE required.

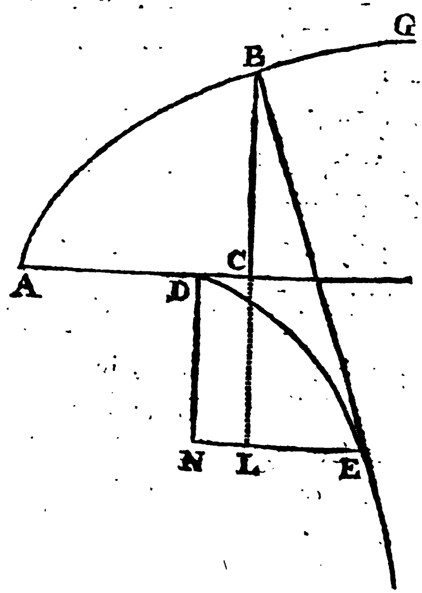
86. *Note*, If the given Involute be *convex* towards its Axis, and x and y increase together, or the *Fluxions* of x and y be both Affirmative; these general Expressions for BL and LE, will be $\frac{\overline{1+\dot{y}^2}}{-\ddot{y}}$, and $\dot{y}\times\frac{\overline{1+\dot{y}^2}}{-\ddot{y}}$, respectively; wherein, the negative Sign, only shews, that the Points L and E must be taken on the Concave Side of the involute Curve, that is, on the other Side of it with Regard to x and y.

EXAMPLE I.

87. *To find the Nature of that Curve by whose Evolution the common Parabola* ABG *is described.*

PUT

PUT AC $= x$, CB $= y$; and DN $= u$, NE $= v$. Now, by *Article* 74. $\dot{y} = \frac{a}{2 \times \overline{ax}|^{\frac{1}{2}}}$, $\dot{y}^2 =$



$\frac{a}{4x}$, and $\ddot{y} = \frac{a^2}{4 \times \overline{ax}|^{\frac{3}{2}}}$; which Values of $\dot{y}$, $\dot{y}^2$, and $\ddot{y}$, being ſubſtituted for them, the general Expreſſion for BL, *viz.* $\frac{1 + \dot{y}^2}{\ddot{y}}$ (*Art.* 85.)

$$\text{becomes} = \frac{\overline{1 + \frac{a}{4x}} \times 4 . \overline{ax}|^{\frac{3}{2}}}{a^2} =$$

$\frac{\overline{4x+a}\times\overline{ax}|^{\frac{1}{2}}}{a}$; and that for L E, *viz.* $\dot{y}\times\frac{1+\dot{y}^2}{\ddot{y}}=\dot{y}\times BL=\frac{a}{2\times\overline{ax}|^{\frac{1}{2}}}\times\frac{\overline{4x+a}\times\overline{ax}|^{\frac{1}{2}}}{a}$ $=2x+\frac{1}{2}a$. Hence, $u\ (=BL-BC)=\frac{\overline{4x+a}\times\overline{ax}|^{\frac{1}{2}}}{a}-(y\text{ or})\ \overline{ax}|^{\frac{1}{2}}=\frac{\overline{4x.ax}|^{\frac{1}{2}}}{a}$; and (becauſe by *Art.* 74. the vertical Diſtance $AD=\frac{1}{2}a$,) $v\ (=AC-AD+LE)=x-\frac{1}{2}a+2x+\frac{1}{2}a=3x$. Now, the former of theſe two Equations produces $x^3=\frac{au^2}{16}$; and, the Cube of the latter, divided by 27, is $x^3=\frac{v^3}{27}$: therefore, $\frac{au^2}{16}=\frac{v^3}{27}$, and $\frac{27a}{16}u^2=v^3$; which is the Equation of the Curve DE, expreſſing the Relation between its Abſciſs and Ordinate; and, the Equation of the cubical Parabola (whoſe Parameter is $\frac{27a}{16}$,) being the ſame; therefore, the Evolute DE, is the Semi-cubical Parabola, whoſe Vertex is D.

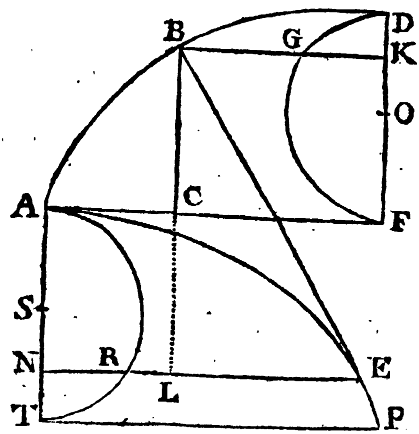
EXAMPLE II.

88. *To find the Nature of the Curve* AEP, *by whoſe Evolution the given Cycloid* ABD *is deſcribed**.

S PUT

* *See in what Manner a* Cycloid *is generated*, Art. 48. Note.

PUT AC$=x$, CB$=y$, Arch FG$=z$, OD or OF$=a$; then (*Art.* 76.) $\dot{y}=\frac{\overline{2ay-y^2}|^{\frac{1}{2}}}{y}$,



$\dot{y}^2=\frac{2ay-y^2}{y^2}$, and $\ddot{y}=\frac{a}{y^2}$: Wherefore, (See *Art.* 85.) BL$=\frac{1+\dot{y}^2}{\ddot{y}}=$

$\frac{1+\frac{2ay-y^2}{y^2}\times y^2}{a}=2y$, and LE$=\dot{y}\times$BL$=$

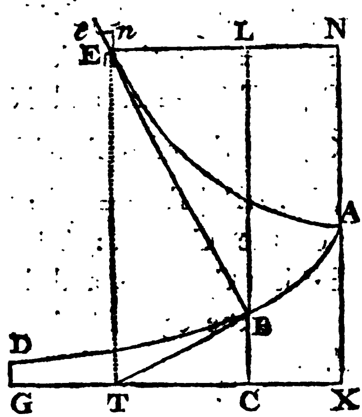
$\frac{\overline{2ay-y^2}|^{\frac{1}{2}}}{y}\times 2y=2\overline{2ay-y^2}|^{\frac{1}{2}}$. Hence, if we put the Abſciſs AN$=u$, and Ordinate NE$=v$, we have $u=$(BL$-$CB$=$) $2y-y=y$, and $v=$(AC+LE$=$) $x+2\times\overline{2ay-y^2}|^{\frac{1}{2}}$

$\overline{2ay-y^2}|^{\frac{1}{2}}$, i. e. (because, *Article* 76. $x = z - \overline{2ay-y^2}|^{\frac{1}{2}}$,) $v = z + \overline{2ay-y^2}|^{\frac{1}{2}}$, or, writing u for y its equal, $v = z + \overline{2au-u^2}|^{\frac{1}{2}}$. Wherefore the evolute Curve AEP is a Cycloid, and equal to the given Cycloid ABD. For, let $AS = ST = a$; then (AN being $= FK$,) $AR = FG = z$, and $NR = \overline{2au-u^2}|^{\frac{1}{2}} = KG$; and therefore $AR + RN = z + \overline{2au-u^2}|^{\frac{1}{2}}$, i.e. $AR + RN = NE$; which is the Property of the Cycloid: Therefore the Evolute AEP is a Cycloid; and, because $AT = FD$, therefore the Cycloids AEP and ABD are equal.

EXAMPLE III.

89. *To find the Nature of the Evolute of the Curve* AD, *whose Tangent*, BT, *is every where equal to the same given Line*, $=a$.

Let BE be the Radius of Curvature at the Point B; then, by *Art.* 79. if a Perpendicular be erected on the Point T, it will pass thro' the Point E; wherefore, when the



Point T coincides with X, i. e. when the Tangent and Ordinate become equal, or, the Points B and A coincide, the Point E will likewise coincide with A: Consequently, the Vertex of the Evolute coincides with that of the Involute. —— Put GX $= b$, GC $= x$, CB $= y$, AN $= u$, NE $= v$; then (*Art.* 79.)

$\dot{y} = \frac{y}{\overline{a^2 - y^2})^{\frac{1}{2}}}$, $\dot{y}^2 = \frac{y^2}{a^2 - y^2}$, and $\ddot{y} = \frac{a^2 y}{\overline{a^2 - y^2})^2}$.

Wherefore, BL (*Art.* 86.) $= \frac{1 + \dot{y}^2}{-\ddot{y}} =$

1 +

$\overline{1+\frac{y^2}{a^2-y^2}} \times -\frac{\overline{a^2-y^2}|^{2}}{a^2y} = \frac{a^2-y^2}{-y}$, and LE $= \dot{y} \times \frac{1+\dot{y}^2}{-\ddot{y}} = \dot{y} \times \text{BL} = \frac{y}{\overline{a^2-y^2}|^{\frac{1}{2}}} \times \frac{a^2-y^2}{-y} = -\overline{a^2-y^2}|^{\frac{1}{2}}$; that is, (because the negative Sign only shews that the Points L and E must be taken on the Concave Side of the Involute Curve AD,) BL $= \frac{a^2-y^2}{y}$, and LE $= \overline{a^2-y^2}|^{\frac{1}{2}}$. Hence we have $u =$ (LB $+$ BC$-$AX$=$) $\frac{a^2-y^2}{y} + y - a$, which gives $y = \frac{a^2}{u+a}$, and therefore $\dot{y} = -\frac{a^2\dot{u}}{\overline{u+a}|^2}$: Also, $v =$ (GX$-$GC$+$CT$=$) $b-x+\overline{a^2-y^2}|^{\frac{1}{2}}$, and therefore $\dot{v} = -\dot{x} - \frac{y\dot{y}}{\overline{a^2-y^2}|^{\frac{1}{2}}} =$ (because, *Article* 38. CT $= \frac{\dot{x}y}{\dot{y}} = \overline{a^2-y^2}|^{\frac{1}{2}}$, or, $\dot{x} = \frac{\dot{y}}{y} \times \overline{a^2-y^2}|^{\frac{1}{2}}$,) $-\frac{\dot{y}}{y} \times \overline{a^2-y^2}|^{\frac{1}{2}} - \frac{y\dot{y}}{\overline{a^2-y^2}|^{\frac{1}{2}}} = \frac{-a^2\dot{y}}{y \times \overline{a^2-y^2}|^{\frac{1}{2}}}$, i. e. (by writing for y and $\dot{y}$ their above Values affected with u and $\dot{u}$,) $\dot{v} = \frac{a\dot{u}}{\overline{u^2+2au}|^{\frac{1}{2}}}$; which is an E-

quation

quation for the Evolute Curve AE; and is also an Equation of the *Catenary* Curve: therefore, the Evolute AE is the *Catenary*.

But, the above Equation of the Evolute, may be found otherwise, thus. Let $En = u'$, and $ne = v'$; then, the Trangles enE and EBT being similar, we have en : nE : : TB : BE, i. e. $v' : u' :: a : \frac{au'}{v'} = BE$,

or (*Article* 6.) $\frac{a\dot{u}}{\dot{v}} = BE$: But, $BE = \overline{\overline{ET}|^2 - \overline{TB}|^2}|^{\frac{1}{2}}$, i. e. (because $ET = NX = u + a$,) $BE = \overline{\overline{u+a}|^2 - a^2}|^{\frac{1}{2}} = \overline{u^2 + 2au}|^{\frac{1}{2}}$: Therefore, $\frac{a\dot{u}}{\dot{v}} = \overline{u^2 + 2au}|^{\frac{1}{2}}$, and $\dot{v} = \frac{a\dot{u}}{\overline{u^2+2au}|^{\frac{1}{2}}}$; as before.

Note. The *Catenary* is a Curve, as ACB,

formed by a flexible Line hanging freely from

from two Points of Suſpenſion, A and B, whether theſe Points be Horizontal or not.

90. THE *Evolute* of a *Spiral*, may be obtained, by finding the Radii of Curvature, in Numbers, at ſeveral Points of the *Involute*; which will give as many Points in the *Evolute* Curve: and then, a Curve Line drawn through the Points ſo found, will be the *Evolute* ſought.

PART

PART II.

CHAP. I.

Of Infinite Series.

AS the Learner may, perhaps, be unacquainted with *Infinite Series*, the Knowledge of which is sometimes absolutely necessary in Order to find the *Fluents* (or flowing Quantities) of *Fluxions* expressed in a fractional Manner, and of such wherein there are Surds or radical Quantities; and, because in some of the following Chapters, the *Fluents* of such fluxional Expressions are to be found; the adding of this Chapter, may therefore not be improper, though it is, in some Measure, foreign to the Business in Hand.

PROB. I.

To reduce fractional Quantities into infinite Series; i. e. to find an infinite Number of Terms, the Sum of which is equal to a given fractional Expression.

EXAMPLE I.

91. *Let it be required to reduce* $\frac{b}{a+x}$ *into an infinite Series.*

PLACE $a+x$ as a Divisor, and b as a Dividend; and divide, as in common Division, till you have 4, 5, 6, or more Terms in the Quotient; (See *the Operation below*;) after which, you may find as many Terms as you please, by considering the Law of the Progression of the Terms already found. Thus, the four first Terms being $\frac{b}{a} - \frac{bx}{a^2} + \frac{bx^2}{a^3} - \frac{bx^3}{a^4}$, the Law of the Continuation of the Division, or of the Series, is plain; for the Quotient consists of an infinite Series of fractional Terms, whose Numerators are the Powers of x (the Indices of which are 1 less than the Numbers of the Terms in which they respectively stand,) multiplied by b; and whose

whose Denominators are the Powers of a, the Indices of which are 1 more than those of the corresponding Numerators; the connective Signs + and − being alternately changed: Therefore the fifth Term will be $+\frac{bx^4}{a^5}$, and the sixth Term $-\frac{bx^5}{a^6}$: And, if an infinite Series, or Number of Terms, be so taken, it will be the exact Quotient of the Division, and consequently exactly equal to the given fractional Expression: But (generally) a few of the first Terms of the Series, are near enough the Truth for any Purpose.

OPERATION.

$$
\begin{array}{l}
a+x)\ b \ \ldots\ \left(\dfrac{b}{a}-\dfrac{bx}{a^2}+\dfrac{bx^2}{a^3}-\dfrac{bx^3}{a^4}+\&c.\right. \\
\qquad b+\dfrac{bx}{a} \\
\hline
\qquad 0-\dfrac{bx}{a} \\
\qquad\quad -\dfrac{bx}{a}-\dfrac{bx^2}{a^2} \\
\hline
\qquad\qquad 0+\dfrac{bx^2}{a^2} \\
\qquad\qquad\quad \dfrac{bx^2}{a^2}+\dfrac{bx^3}{a^3} \\
\hline
\qquad\qquad\quad 0-\dfrac{bx^3}{a^3} \\
\qquad\qquad\qquad -\dfrac{bx^3}{a^3}-\dfrac{bx^4}{a^4} \\
\hline
\qquad\qquad\qquad\quad 0+\dfrac{bx^4}{a^4}
\end{array}
$$

Or, if we put x before a, in the Denominator of the above fractional Expression, i. e. if the Divisor be placed thus, $x+a$, instead of $a+x$; then, the Quotient, or Series, will be $\frac{b}{x}-\frac{ba}{x^2}+\frac{ba^2}{x^3}-\frac{ba^3}{x^4}+\&c.$ Whence

the

the Law of the Continuation of the Series may be obſerved as before.

92. *Note.* THERE will always be as many Quotients, or infinite Series, as there are Terms in the Denominator, or Diviſor; though only one of them true: and to find this, you muſt always place the greateſt Terms in the Diviſor and Dividend firſt; i. e. if in the above Example, for Inſtance, a be greater than x, then a muſt be the firſt Term in the Diviſor, and $\frac{b}{a} - \frac{bx}{a^2} + \frac{bx^2}{a^3} - \frac{bx^3}{a^4} +$ &c. will be the true Series: But, if x be greater than a, then x muſt be the firſt Term in the Diviſor, and $\frac{b}{x} - \frac{ba}{x^2} + \frac{ba^2}{x^3} - \frac{ba^3}{x^4} +$ &c. will be the true Series; the other, then, being a diverging one; and conſequently, the farther you go in the Series, the farther it will be from the Truth.

EXAMPLE II.

93. *Let it be required to throw* $\frac{a^2}{a^2+2ax+x^2}$ *into an infinite Series; ſuppoſing* a^2 *to be greater than* $2ax$ *or* x^2.

OPERA-

OPERATION.

$$
\begin{array}{l}
a^2+2ax+x^2)\ a^2 \ldots\ldots \left(1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^3}+\&c.\right. \\
\qquad\qquad\qquad \underline{a^2+2ax+x^2} \\
\qquad\qquad\qquad 0-2ax-x^2 \\
\qquad\qquad\qquad \underline{-2ax-4x^2-\frac{2x^3}{a}} \\
\qquad\qquad\qquad\qquad 0+3x^2+\frac{2x^3}{a} \\
\qquad\qquad\qquad\qquad \underline{3x^2+\frac{6x^3}{a}+\frac{3x^4}{a^2}} \\
\qquad\qquad\qquad\qquad 0-\frac{4x^3}{a}-\frac{3x^4}{a^2} \\
\qquad\qquad\qquad\qquad \underline{-\frac{4x^3}{a}-\frac{8x^4}{a^2}-\frac{4x^5}{a^3}} \\
\qquad\qquad\qquad\qquad\qquad 0+\frac{5x^4}{a^2}+\frac{4x^5}{a^3}
\end{array}
$$

Now, from these four Terms, it is easy to see, that the Law of Continuation is such, that the Numerators, are the Powers of x, whose Exponents are 1 less than the Numbers of the Terms to which they respectively belong,

belong, multiplied by the said Numbers; and the Denominators, the Powers of a, whose Exponents are the same with those of the corresponding Numerators; the Signs being changed alternately: So that the fifth Term is $+\frac{5x^4}{a^4}$, the sixth Term is $-\frac{6x^5}{a^5}$, and so on.

PROB. II.

To reduce a compound Surd Quantity into an infinite Series; i. e. to free a compound Expression from Surds, by throwing it into an infinite Series.

EXAMPLE I.

94. *Let it be required to throw $\sqrt{a^2+x^2}$, i. e. $\overline{a^2+x^2}|^{\frac{1}{2}}$, into an infinite Series.*

Note. IN order to have a true Series, the greatest Term must be placed first, as in *Prob.* 1. Therefore, supposing a^2 to be greater than x^2,

TAKE the Square Root of a^2, which is a, for the *first* Term of the Root or Series, (See *the Operation below*;) then, this squared and subtracted from a^2+x^2, leaves $+x^2$; and this

this Remainder divided (as in the common Extraction of the Square Root,) by the double of the first Term, *viz.* by $2a$, gives $+ \frac{x^2}{2a}$ for the *second* Term of the Root; which, together with the double of the first Term, being multiplied by $\frac{x^2}{2a}$ the said second Term, gives $x^2 + \frac{x^4}{4a^2}$, and this subtracted from x^2 leaves $-\frac{x^4}{4a^2}$, which divided by the double of the two first Terms in the Root, *viz.* by $2a + \frac{x^2}{a}$, gives $-\frac{x^4}{8a^3}$ for the *third* Term of the Root; which, together with the double of the two first Terms, *viz.* $2a + \frac{x^2}{a}$, being multiplied by $-\frac{x^4}{8a^3}$ the said third Term, gives $-\frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{x^8}{64a^6}$, and this subtracted from $-\frac{x^4}{4a^2}$ leaves $\frac{x^6}{8a^4} - \frac{x^8}{64a^6}$, which divided by the double of the three first Terms of the Root already found,

viz.

viz. by $2a + \frac{x^2}{a} - \frac{x^4}{4a^3}$, gives $+ \frac{x^6}{16a^5}$ for the *fourth* Term in the Root. And, after the ſame Manner, may be found any Number of Terms in the Root: and, when the Law of the Progreſſion, or continuation of the Series, is diſcovered, the Terms may be continued on at Pleaſure.

OPERATION.

$$
\begin{array}{rl}
 & a^2 + x^2 \left(a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} \right. \\
 & a^2 \\
\hline
2a) & 0 + x^2 \\
 & x^2 + \frac{x^4}{4a^2} \\
\hline
2a + \frac{x^2}{a}) & \ldots 0 - \frac{x^4}{4a^2} \\
 & - \frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{x^8}{64a^6} \\
\hline
2a + \frac{x^2}{a} - \frac{x^4}{4a^3}) & \ldots\ldots 0 + \frac{x^6}{8a^4} - \frac{x^8}{64a^6}
\end{array}
$$

EXAMPLE II.

95. *Let it be required to throw* $\overline{x-x^2}|^{\frac{1}{2}}$ *into an infinite Series.*

OPERATION.

$$x-x^2\ \left(x^{\frac{1}{2}}-\frac{1}{2}x^{\frac{3}{2}}-\frac{1}{8}x^{\frac{5}{2}}-\frac{1}{16}x^{\frac{7}{2}}-\frac{5}{128}x^{\frac{9}{2}}-\text{\&c.}\right.$$

$$\begin{array}{rl}
 & x \\
2x^{\frac{1}{2}}) & \overline{0-x^2} \\
 & \underline{-x^2+\frac{1}{4}x^3} \\
2x^{\frac{1}{2}}-x^{\frac{3}{2}})\ \ldots & 0-\frac{1}{4}x^3 \\
 & \underline{-\frac{1}{4}x^3+\frac{1}{8}x^4+\frac{1}{64}x^5} \\
2x^{\frac{1}{2}}-x^{\frac{3}{2}}-\frac{1}{4}x^{\frac{5}{2}})\ \ldots & 0-\frac{1}{8}x^4-\frac{1}{64}x^5 \\
 & \underline{-\frac{1}{8}x^4+\frac{1}{16}x^5+\frac{1}{64}x^6+\frac{1}{256}x^7} \\
2x^{\frac{1}{2}}-x^{\frac{3}{2}}-\frac{1}{4}x^{\frac{5}{2}}-\frac{1}{8}x^{\frac{7}{2}})\ \ldots & 0-\frac{5}{64}x^5-\frac{1}{64}x^6-\frac{1}{256}x^7
\end{array}$$

So

So that $\overline{x-x^2}|^{\frac{1}{2}}$ is $= x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}} - \frac{5}{128}x^{\frac{9}{2}}$ — &c.

AND, after the ſame Manner, may any ſuch common ſurd Quantity be reduced into an infinite Series: But, with much greater Eaſe and Expedition,

96. ALL Sorts of fractional and ſurd Quantities may be reduced into infinite Series, by a moſt curious and excellent Theorem, invented for that purpoſe by the Great and Illuſtrious Sir *Iſaac Newton*, called his

UNIVERSAL THEOREM;

which is this, *viz.* $\overline{P+PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + \frac{m-4n}{5n}EQ +$ &c. Wherein, it muſt be obſerved, that, $P+PQ$ repreſents the Quantity, whoſe Root, Dimenſion, or Root of the Dimenſion, is required to be thrown into an infinite Series; P the firſt Term of that Quantity, which muſt be the greateſt; Q the reſt of the Terms divided by

by the first; $\frac{m}{n}$ the numerical Index of the Dimension or Power of $P+PQ$, whether that Power be Affirmative or Negative, Integral or Fractional; and A,B,C,D,E, &c. the Terms in the Series or Quotient already found; i. e. $A=P^{\frac{m}{n}}$, $B=\frac{m}{n}AQ$, $C=\frac{m-n}{2n}BQ$, $D=\frac{m-2n}{3n}CQ$, $E=\frac{m-3n}{4n}DQ$; or, A = the first Term, B = the second Term, C = the third Term, D = the fourth Term, E = the fifth Term, &c. ——The following Examples, will explain, and shew the great Use of this noble Theorem.

EXAMPLE I.

97. *Let it be required to throw $\overline{a^2+x^2}|^{\frac{1}{2}}$ into an infinite Series; a^2 being supposed greater than x^2.*

HERE, $P=a^2$, $Q=\frac{x^2}{a^2}$, $m=1$, $n=2$, $A=P^{\frac{m}{n}}=a$, $B=\frac{m}{n}AQ=\frac{x^2}{2a}$, $C=\frac{m-n}{2n}\times BQ=-\frac{x^4}{8a^3}$, $D=\frac{m-2n}{3n}CQ=\frac{x^6}{16a^5}$,

E=

$E = \frac{m-3n}{4n}DQ = -\frac{5x^8}{128a^7}$; &c. therefore $\overline{a^2+x^2}|^{\frac{1}{2}} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$, &c.

EXAMPLE II.

98. *Let it be required to put $\overline{x-x^2}|^{\frac{1}{2}}$ into an infinite Series.*

HERE, $P = x$, $Q = -\frac{x^2}{x} = -x$, $m = 1$, $n = 2$, $A = x^{\frac{1}{2}}$, $B = -\frac{1}{2} \times x^{\frac{3}{2}}$, $C = -\frac{1}{8}x^{\frac{5}{2}}$, $D = -\frac{1}{16}x^{\frac{7}{2}}$, &c. therefore $\overline{x-x^2}|^{\frac{1}{2}} = x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}}$, &c.

EXAMPLE III.

99. *Let it be required to put $\frac{b}{a+x}$ into an infinite Series; x being less than a.*

$\frac{b}{a+x}$

$\frac{b}{a+x}$ is $= b \times \frac{1}{a+x} = b \times \overline{a+x}|^{-1}$; and $P = a$, $Q = \frac{x}{a}$, $m = 1$, $n = -1$, $A = a^{-1} = \frac{1}{a}$, $B = -\frac{x}{a^2}$, $C = \frac{x^2}{a^3}$, $D = -\frac{x^3}{a^4}$, $E = \frac{x^4}{a^5}$, &c. therefore $b \times \overline{a+x}|^{-1} = b \times : \frac{1}{a} - \frac{x}{a^2} + \frac{x^2}{a^3} - \frac{x^3}{a^4} + \frac{x^4}{a^5}$, &c. that is, $\frac{b}{a+x} = \frac{b}{a} - \frac{bx}{a^2} + \frac{bx^2}{a^3} - \frac{bx^3}{a^4} + \frac{bx^4}{a^5}$, &c.

Example IV.

100. *Let it be required to put* $\frac{a^2}{\overline{a+x}|^2}$ *into an infinite Series; where* a *is supposed greater than* x.

$\frac{a^2}{\overline{a+x}|^2}$ is $= a^2 \times \overline{a+x}|^{-2}$; and $P = a$, $Q = \frac{x}{a}$, $m = -2$, $n = 1$, $A = a^{-2} = \frac{1}{a^2}$, $B = -\frac{2x}{a^3}$, $C = \frac{3x^2}{a^4}$, $D = -\frac{4x^3}{a^5}$, $E =$

$\frac{5x^4}{a^6}$, &c. therefore $a^2 \times \overline{a+x}|^{-2} = a^2 \times :$ $\frac{1}{a^2} - \frac{2x}{a^3} + \frac{3x^2}{a^4} - \frac{4x^3}{a^5} + \frac{5x^4}{a^6}$, &c. that is, $\frac{a^2}{\overline{a+x}|^2} = 1 - \frac{2x}{a} + \frac{3x^2}{a^2} - \frac{4x^3}{a^3} + \frac{5x^4}{a^4}$, &c.

EXAMPLE V.

101. *Let it be required to put* $\frac{1}{\overline{a^2-x^2}|^{\frac{1}{2}}}$, i. e. $\overline{a^2-x^2}|^{-\frac{1}{2}}$, *into an infinite Series.*

HERE $P = a^2$, $Q = -\frac{x^2}{a^2}$, $m = -1$, $n = 2$; and $A = \overline{a^2}|^{-\frac{1}{2}} = \frac{1}{a}$, $B = -\frac{1}{2} \times \frac{1}{a} \times -\frac{x^2}{a^2} = \frac{x^2}{2a^3}$, $C = -\frac{3}{4} \times \frac{x^2}{2a^3} \times -\frac{x^2}{a^2} = \frac{3x^4}{8a^5}$, $D = -\frac{5}{6} \times \frac{3x^4}{8a^5} \times -\frac{x^2}{a^2} = \frac{15x^6}{48a^7}$, &c. therefore $\overline{a^2-x^2}|^{-\frac{1}{2}}$ is $= \frac{1}{a} + \frac{x^2}{2a^3} + \frac{3x^4}{8a^5} + \frac{15x^6}{48a^7}$, &c.

102. BUT,

102. BUT, we may often find the Series anſwering to a propoſed Quantity, by the following *Univerſal Theorem*, viz. $\overline{P+PQ}|^{\frac{m}{n}}$ $= P^{\frac{m}{n}} \times : 1 + \frac{m}{n}Q + \frac{m}{n} \times \frac{m-n}{2n}Q^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}Q^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n}Q^4 +$ &c. (which indeed is the ſame as the foregoing, though differently expreſſed,) with ſtill more Eaſe and Expedition; for herein, no previous Deduction is required; and both the Numerators and Denominators of the Fractions $\frac{m}{n}$, $\frac{m-n}{2n}$, $\frac{m-2n}{3n}$, $\frac{m-3n}{4n}$, &c. are Series of Numbers in Arithmetical Progreſſion, which have the ſame common Difference n. This will appear by the following Examples. — But *Note*, The former Theorem is, in general, beſt adapted to ſhew the *Law* of the Series.

EXAMPLE I.

103. *Let* $\overline{x-x^2}|^{\frac{1}{2}}$ *be required to be put into an infinite Series.*

HERE

HERE $P = x$, $Q = -\frac{x^2}{x} = -x$, $m = 1$, $n = 2$; therefore $\overline{x - x^2}|^{\frac{1}{2}} = x^{\frac{1}{2}} \times : 1 - \frac{1}{2}x + \frac{1 \times -1}{2 \times 4}x^2 - \frac{1 \times -1 \times -3}{2 \times 4 \times 6}x^3 + \frac{1 \times -1 \times -3 \times -5}{2 \times 4 \times 6 \times 8} \times x^4 -$ &c. i. e. $= x^{\frac{1}{2}} \times : 1 - \frac{1}{2}x - \frac{1}{2.4}x^2 - \frac{3}{2.4.6.}x^3 - \frac{3.5}{2.4.6.8}x^4 -$ &c. or, $x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{2} - \frac{x^{\frac{5}{2}}}{2.4} - \frac{3.x^{\frac{7}{2}}}{2.4.6} - \frac{3.5.x^{\frac{9}{2}}}{2.4.6.8} -$ &c.

EXAMPLE II.

104. *Let* $\overline{a + x}|^{\frac{5}{3}}$ *be required to be put into an infinite Series; supposing a to be more than x.*

HERE $P = a$, $Q = \frac{x}{a}$, $m = 5$, $n = 3$; therefore $\overline{a + x}|^{\frac{5}{3}} = a^{\frac{5}{3}} \times : 1 + \frac{5}{3} \times \frac{x}{a} + \frac{5.2}{3.6} \times \frac{x^2}{a^2} + \frac{5.2.-1}{3.6.9} \times \frac{x^3}{a^3} + \frac{5.2.-1.-4}{3.6.9.12} \times \frac{x^4}{a^4} +$ &c. i.e. $= a^{\frac{5}{3}} + \frac{5.a^{\frac{2}{3}}x}{3} + \frac{5.2.x^2}{4.6.a^{\frac{1}{3}}} - \frac{5.2.x^3}{3.6.9.a^{\frac{4}{3}}} + \frac{5.2.4.x^4}{3.6.9.12.a^{\frac{7}{3}}}$, &c.

EXAMPLE III.

105. *Let* $\frac{b}{a+x}$, i. e. $b\times\overline{a+x}|^{-1}$, *be required to be put into an infinite Series; supposing a to be greater than x.*

HERE $P=a$, $Q=\frac{x}{a}$, $m=1$, $n=-1$; and therefore $\frac{m}{n}$, $\frac{m}{n}\times\frac{m-n}{2n}$, $\frac{m}{n}\times\frac{m-n}{2n}\times\frac{m-2n}{3n}$, $\frac{m}{n}\times\frac{m-n}{2n}\times\frac{m-2n}{3n}\times\frac{m-3n}{4n}$, &c. will be -1 and $+1$ alternately; and consequently $b\times\overline{a+x}|^{-1}$ is $=b\times a^{-1}\times:1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+\frac{x^4}{a^4}-$ &c. i.e. $\frac{b}{a+x}=\frac{b}{a}-\frac{bx}{a^2}+\frac{bx^2}{a^3}-\frac{bx^3}{a^4}+\frac{bx^4}{a^5}-$ &c.

EXAMPLE IV.

106. *Let* $\overline{1-x}|^{\frac{1}{4}}$ *be required to be put into an infinite Series.*

HERE

HERE $P=1$, $Q=-x$, $m=1$, $n=4$; therefore $\overline{1-x}|^{\frac{1}{4}} = 1 - \frac{1}{4}x + \frac{1.-3}{4.8}x^2 - \frac{1.-3.-7}{4.8.12}x^3 + \frac{1.-3.-7.-11}{4.8.12.16}x^4 -$ &c. that is, $= 1 - \frac{1}{4}x - \frac{3}{4.8}x^2 - \frac{3.7}{4.8.12}x^3 - \frac{3.7.11}{4.8.12.16}x^4 -$ &c.

CHAP. II.

Of finding the FLUENT *of a given* FLUXION.

107. THE Business of the *Direct Method of Fluxions*, being to find the *Fluxion* of a given *Fluent*, or, the *Velocity* with which a *variable Quantity* flows at any Point or Term assigned: So, the Business of the *Inverse Method of Fluxions*, is to determine the *variable Quantity*, or *Fluent*, from that *Velocity* or *Fluxion* being given. And this, in general, may be done by the following *Rules*; these being derived from those delivered in *Part* 1. *Chap.* 2.

Rule I.

108. *To find the* Fluent *of a* Simple Fluxion; *or, of that wherein there is no variable Quantity, and but one Fluxional Letter.*

Substitute the variable or flowing Letter for its Fluxion; and you will have the Fluent required.

Thus, the *Fluent* of $a\dot{x}$ is $=ax$.

Rule II.

109. *To find the* Fluent *of a fluxional Expression which consists of the Products of two or more variable (or variable and invariable) Quantities drawn into their* Fluxions; *i. e. which consists of the* Fluxion *of each Quantity, multiplied into the other or Product of the rest of the Quantities: as* $\dot{x}y+x\dot{y}$, *or* $\dot{x}yz+x\dot{y}z+xy\dot{z}$, *or* $a\dot{x}y+ax\dot{y}$.

Multiply the variable and (if any) invariable Quantities together; and the Product is the *Fluent* sought.

Thus, the *Fluent* of $\dot{x}y+x\dot{y}$ is $=xy$; and the *Fluent* of $\dot{x}yz+x\dot{y}z+xy\dot{z}$ is $=xyz$: So likewise, the *Fluent* of $a\dot{x}y+ax\dot{y}$ is $=axy$; and the *Fluent* of $a\dot{x}yz+ax\dot{y}z+axy\dot{z}$ is $=axyz$.

Rule

RULE III.

110. *To find the* Fluent *of a fluxional Expression which consists of the* Fluxion *of a variable Quantity drawn into any Power of that Quantity contained any Number of Times; as* $2x^2\dot{x}$.

STRIKE out the fluxional Letter; add 1 to the Index of the Power of the variable Quantity; and divide the Coefficient by the Index thus increased: and then you will have the *Fluent* sought.

Thus, the *Fluent* of $2x^2\dot{x}$ is $= \frac{2}{3}x^3$; the *Fluent* of $-\frac{\dot{x}}{x^2}$, i. e. of $-x^{-2}\dot{x}$, is $= -x^{-2+1}$ divided by $-2+1$, that is, $= \frac{-x^{-1}}{-1}$, or x^{-1}, or $\frac{1}{x}$. And, Universally, the *Fluent* of $mx^{m-1}\dot{x}$ is $= x^m$; the *Fluent* of $\frac{m}{n}x^{\frac{m}{n}-1}\dot{x}$ is $= x^{\frac{m}{n}}$; and the *Fluent* of $ax^{\frac{m}{n}}\dot{x}$ is $= \frac{an}{m+n}x^{\frac{m+n}{n}}$.

111. *Or*, from Examples given in the converse of this *Rule*, (*Article* 16.) it appears, that, in fluxional Expressions, like $\overline{x+x^2}|^2 \times$

$3\dot{x}+$

$\overline{3\dot{x}+6x\dot{x}}$, where ths fluxionary Part is in any invariable Ratio to the *Fluxion* of the Quantity under the Vinculum; the *Fluents* of ſuch Expreſſions may be found in a different Manner; *viz.* by multiplying the whole Expreſſion by the Root or Quantity under the Vinculum, and dividing the Product by the *Fluxion* of the ſaid Root drawn into the Index of the Power added to 1.

Thus, the *Fluent* of $\overline{x+x^2}|^2 \times \overline{3\dot{x}+6x\dot{x}}$ is $=$

$$\frac{\overline{x+x^2}|^2 \times \overline{3\dot{x}+6x\dot{x}} \times \overline{x+x^2}}{\overline{\dot{x}+2x\dot{x}} \times \overline{2+1}} =$$

$$\frac{\overline{x+x^2}|^2 \times \overline{3\dot{x}+6x\dot{x}} \times \overline{x+x^2}}{3\dot{x}+6x\dot{x}} = \overline{x+x^2}|^2 \times \overline{x+x^2}$$

$= \overline{x+x^2}|^3$; and the *Fluent* of $\frac{-2\dot{x}}{\overline{a+x}|^3}$, or $-\overline{a+x}|^{-3} \times 2\dot{x}$ is $= \frac{-\overline{a+x}|^{-3} \times 2\dot{x} \times \overline{a+x}}{\dot{x} \times \overline{-3+1}}$

$= \overline{a+x}|^{-3} \times \overline{a+x} = \overline{a+x}|^{-2} = \frac{1}{\overline{a+x}|^2}$.

RULE IV.

112. *To find the* Fluent *of a fluxional Expreſſion conſiſting of different fluxionary Terms connected together by the Signs* $+$ *and* $-$.

FIND

FIND the ſeparate Fluents of the different Terms by the preceding *Rules*; which connected together by the Signs of their reſpective *Fluxions*, will be the *Fluent* of the Expreſſion required.

Thus, the *Fluent* of $a\dot{x}+2y\dot{y}-\dot{z}$ is $= ax+y^2-z$; the *Fluent* of $\dot{x}-a^2\dot{y}+\dot{x}y+x\dot{y}$ is $= x-a^2y+xy$; and the *Fluent* of $2ax\dot{x}+\dot{x}-.\frac{\dot{x}y-x\dot{y}}{y^2}$ is $= ax^2+x-\frac{x}{y}$.

RULE V.

113. *To find the* Fluent *of a compounded fluxional Expreſſion like* $\frac{b}{a+x}\dot{x}$, *or* $\overline{x-x^2}|^{\frac{1}{2}}\dot{x}$.

THROW the Expreſſion into an infinite Series, as taught in the laſt Chapter; and find the *Fluent* of the Series by the foregoing *Rules*: and it will be the *Fluent* ſought.

Thus, to find the *Fluent* of $\frac{b}{a+x}\dot{x}$; I throw it into a Series; which (*Article* 99.) is $=\frac{b}{a}\dot{x}-\frac{bx}{a^2}\dot{x}+\frac{bx^2}{a^3}\dot{x}-\frac{bx^3}{a^4}\dot{x}+\frac{bx^4}{a^5}\dot{x}$, &c. and then find the Fluent of this Series; which (See *Art.* 110 and 112.) is $=\frac{bx}{a}-$

bx^2

$\frac{bx^2}{2a^2}+\frac{bx^3}{3a^3}-\frac{bx^4}{4a^4}+\frac{bx^5}{5a^5}$, &c. and this is the Fluent of $\frac{b}{a+x}\dot{x}$ required. — And to find the *Fluent* of $\overline{x-x^2}|^{\frac{1}{2}}\dot{x}$; the Expression must be thrown into a Series, which (*Article* 98.) is $= x^{\frac{1}{2}}\dot{x} - \frac{1}{2}x^{\frac{3}{2}}\dot{x} - \frac{1}{8}x^{\frac{5}{2}}\dot{x} - \frac{1}{16}x^{\frac{7}{2}}\dot{x}$, &c. and the Fluent of this Series (*Art.* 110 and 112.) is $= \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}} - \frac{1}{72}x^{\frac{9}{2}}$, &c. which is the Fluent of $\overline{x-x^2}|^{\frac{1}{2}}\dot{x}$ required.

114. But, it is not always Necessary to throw an Expression like $\frac{b\dot{x}}{a+x}$, or $\frac{\dot{x}}{\overline{a^2+x^2}|^{\frac{1}{2}}}$, &c. into an *infinite Series* in order to find the *Fluent*, since it may often be obtained otherwise by means of *Hyperbolic Logarithms.* For, (*Art.* 17.) the Fluxion of the Hyperbolic Logarithm of any Quantity, being equal to the Fluxion of that Quantity divided by the Quantity itself; therefore, the *Fluent* of $b \times \frac{\dot{x}}{a+x}$ is $= b \times$ Hyperbolic Logarithm of $\overline{a+x}$; the Fluxion of $a+x$, divided by $a+x$, being

being $= \frac{\dot{x}}{a+x}$. And thus, the *Fluent* of $\frac{\dot{x}}{\overline{a^2+x^2}|^{\frac{1}{2}}}$ is = the Hyperbolic Logarithm of $x+\overline{a^2+x^2}|^{\frac{1}{2}}$; for the Fluxion of $x+\overline{a^2+x^2}|^{\frac{1}{2}}$ being divided by $x + \overline{a^2+x^2}|^{\frac{1}{2}}$, gives $\frac{\dot{x}}{\overline{a^2+x^2}|^{\frac{1}{2}}}$. So likewise, the *Fluent* of $\frac{2a\dot{x}}{a^2-x^2}$ is = the Hyperbolic Logarithm of $\frac{a+x}{a-x}$; for the Fluxion of $\frac{a+x}{a-x}$, divided by $\frac{a+x}{a-x}$, is $= \frac{2a\dot{x}}{a^2-x^2}$. And, the *Fluent* of $\frac{\dot{x}}{\overline{2ax+x^2}|^{\frac{1}{2}}}$ is = the Hyp. Log. of $a+x+\overline{2ax+x^2}|^{\frac{1}{2}}$ (See *Article* 17.) Alſo, the *Fluent* of $\frac{2a\dot{x}}{x\times\overline{a^2+x^2}|^{\frac{1}{2}}}$ is = the Hyperbolic Logarithm of $\frac{a-\overline{a^2+x^2}|^{\frac{1}{2}}}{a+\overline{a^2+x^2}|^{\frac{1}{2}}}$; for, if by $\frac{a-\overline{a^2+x^2}|^{\frac{1}{2}}}{a+\overline{a^2+x^2}|^{\frac{1}{2}}}$ we divide its Fluxion, the Quotient will become $= \frac{2a\dot{x}}{x\times\overline{a^2+x^2}|^{\frac{1}{2}}}$.

SCHOLIUM.

115. Though the *Fluxion* of any *Fluent*, how much ſoever compounded it be with invariable Quantities, may be accurately found; yet, the *Fluents* of ſuch compounded fluxional Expreſſions cannot always be had in finite Terms. — And, though no *Fluent* can have more than *one Fluxion*; yet a *Fluxion* may have an *infinite Number* of *Fluents*: Thus, for Example, the *Fluent* of $\dot{x}$ may be either x, or $x \pm a$; wherein a repreſents any invariable Quantity whatſoever: And, to find a, when it muſt be added to, or taken from, the *Fluent* x, is called *correcting* the *Fluent*. Now, to effect this, in any fluxional Equation; after having obtained the *Fluent* of each Side, by the foregoing *Rules*; make the variable Letter in either of them Vaniſh or equal to Nothing; and ſubſtitute for the variable Letter in the other, ſuch a determinate or invariable Value as it is then known to have: or, for the variable Quantities, write ſuch invariable or fixt Values as they are reſpectively known to have at any particular Point or Term. Then, if we ſubtract the Sides of this new Equation from the correſponding *Fluents* before found; theſe remaining

ing *Fluents*, will be *contemporary*, or always equal to each other; and consequently, we shall then have the *correct Fluent* sought determined. This Affair, may, perhaps, be better understood by giving the following Examples.

Example 1.

To find the Fluent *of* $\dot{y}=2x\dot{x}$.

The Fluent of this (by *Article* 110.) is $y=x^2$. Now, when $y=0$, if $x=0$; then, $y-0=x^2-0$, or $y=x^2$; and therefore, the Fluent first found, needs no Correction.

Example 2.

To find the Fluent *of* $a\dot{x}-2x\dot{x}=2y\dot{y}$

The Fluent of this (*Article* 110.) is $ax-x^2=y^2$. Now, to know whether this Fluent needs Correction; when $x=0$, if y be likewise $=0$; then, it is evident, both Sides of the fluential Equation entirely vanish; and therefore, it needs no Correction. Or, when $y=0$, if x be $=a$; then, substituting 0 for y, and a for x, the Equation will become $a^2-a^2=0$; and therefore, neither on this supposition does the Fluent need Correction.

Example 3.

To find the Fluent *of* $\dot{y} = x^2 \dot{x}$.

The Fluent of this (*Article* 110.) is $y = \frac{1}{3} x^3$. Now, if x be $= a$ when y is $= 0$; then, by writing in this Fluent, 0 for y, and a for x, it will be $0 = \frac{1}{3} a^3$; therefore $\frac{1}{3} x^3$ is always $\frac{1}{3} a^3$ more than y; and consequently, the Fluent corrected is $y = \frac{1}{3} x^3 - \frac{1}{3} a^3$.

Example 4.

To find the Fluent *of* $\dot{y} = -\overline{a+x}^{-3} \times 2\dot{x}$.

The Fluent of this (*Article* 110.) is $y = \frac{-\overline{a+x}^{-2} \times 2}{-2} = \overline{a+x}^{-2} = \frac{1}{\overline{a+x}^{2}}$. Now, when $y = 0$, if $x = 0$; this Fluent will then become $0 = \frac{1}{a^2}$; and therefore, y is always

less

leſs than $\frac{1}{\overline{a+x}|^2}$, by $\frac{1}{a^2}$: So that, the contemporary Fluents will be $y = \frac{1}{\overline{a+x}|^2} - \frac{1}{a^2}$.

Example 5.

To find the Fluent *of* $\dot{z} = a^2 \times \frac{\dot{x}}{\overline{a^2+x^2}|^{\frac{1}{2}}}$.

The *Fluxion* of the hyp. Log. of $\overline{x+\overline{a^2+x^2}|^{\frac{1}{2}}}$ is $= \frac{\dot{x}}{\overline{a^2+x^2}|^{\frac{1}{2}}}$ (*Art.* 17. or 114.) therefore, the *Fluent* of $\dot{z} = a^2 \times \frac{\dot{x}}{\overline{2^2+x^2}|^{\frac{1}{2}}}$, is $z = a^2 \times$ hyp. Log. of $\overline{x+\overline{a^2+x^2})^{\frac{1}{2}}}$. Now, when $z = 0$, if x be likewiſe $= 0$, this Fluent will then become $0 = a^2 \times$ hyp. Log. of a; which ſubtracted from the ſaid Fluent, makes the correct Fluent, or true Value of $z = a^2 \times$ hyp. Log. of $\overline{x+\overline{a^2+x^2}|^{\frac{1}{2}}} - a^2 \times$ hyp. Log. of a (which by the Nature of Logarithms is) $= a^2 \times$ hyp. Log. of $\frac{x+\overline{a^2+x^2}|^{\frac{1}{2}}}{a}$.

Exam-

Example 6.

To find the correct Fluent *of* $a\dot{x} - bx\dot{x} = y\dot{y}$.

The Fluent of this (*Art.* 110.) is $ax - \frac{1}{2}bx^2 = \frac{1}{2}y^2$. Now, if $x = c$ when $y = d$; then, substituting c for x and d for y, the Equation will be $ac - \frac{1}{2}bc^2 = \frac{1}{2}d^2$; and therefore, by subtracting the corresponding Sides of this Equation from the above, we have the correct or contemporary Fluents $ax - \frac{1}{2}bx^2 - ac + \frac{1}{2}bc^2 = \frac{1}{2}y^2 - \frac{1}{2}d^2$.

CHAP.

CHAP. III.

Of finding the Length of a Curve Line.

Fig. 1.

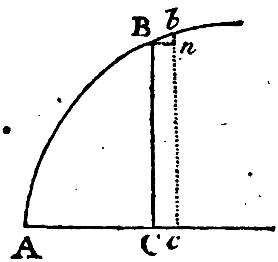
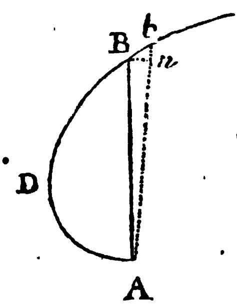


Fig. 2.



116 IN Curves referred to an Axis, (*Fig.* 1.) let *bc* be ſuppoſed indefinitely near and parallel to the Ordinate BC, and *Bn* equal and parallel to C*c* the Increment of the Abſciſs AC. And, in Curves referred to a fixt or central Point, (*Fig.* 2.) let *b*A be ſup-

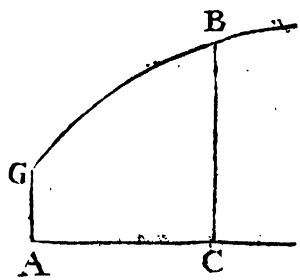
supposed indefinitely near to BA, and the indefinitely little circular Arch Bn be described with the Ordinate or Radius AB. Put AC (*Fig.* 1.) $=x$, CB$=y$, Curve AB$=z$; B$n=x'$, $nb=y'$, and B$b=z'$: then, (the Increment Bb being considered as a little right Line,) by 47 E. 1. B$b=\overline{\mathrm{B}n|^2+nb|^2}|^{\frac{1}{2}}$, i. e. $z'=\overline{x'^2+y'^2}|^{\frac{1}{2}}$, or (*Art.* 6.) $\dot{z}=\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}$. Put the Ordinate AB (*Fig.* 2.) $=y$, Curve ADB$=z$; B$n=x'$, $nb=y'$, and B$b=z'$: then, (because Bb may be considered as an indefinitely small right Line, and Bn as a little right Line perpendicular to Ab,) as before, $z'=\overline{x'^2+y'^2}|^{\frac{1}{2}}$, or $\dot{z}=\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}$: And this is a general Expression for the *Fluxion* of the Length of any Curve-Line whatsoever *. Now, by Help of the Equation or Properties of the given Curve whose Length is required, we may find the Value of $\dot{x}^2$ in Terms of $\dot{y}^2$, or of $\dot{y}^2$ in Terms of $\dot{x}^2$; and then, by Substitution, $\dot{x}^2$ or $\dot{y}^2$ in this general Expression, will be exterminated: and, by finding the *Fluent* of the resulting Equation, we shall have the Value of z, or the Length of the Curve-Line required.

EXAM-

* The same Expression may be derived without the Help of *Increments*, from *Art.* 37. and 53.

EXAMPLE I.

117. *To find the Length of the Curve* GB $=z$; *whose Equation (putting the given Line* AG $=\frac{2}{3}a$, AC $=x$, and CB $=y$,) is $2\times\overline{a^2+x^2}|^{\frac{3}{2}}=3a^2y$.



THE *Fluxion* of this Equation (*Art.* 16.) is $3\times\overline{a^2+x^2}|^{\frac{1}{2}}\times 2x\dot{x}=3a^2\dot{y}$; therefore,

$$\dot{y}=\frac{2x\dot{x}}{3a^2}\times 3\times\overline{a^2+x^2}|^{\frac{1}{2}}=\frac{2x\dot{x}}{a^2}\times\overline{a^2+x^2}|^{\frac{1}{2}},$$

and $\dot{y}^2=\frac{4x^2\dot{x}^2}{a^4}\overline{a^2+x^2}=\frac{4a^2x^2\dot{x}^2+4x^4\dot{x}^2}{a^4}$;

which substituted for $\dot{y}^2$, makes $\dot{z}=\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}$ (*Art.* 116.) $=\left.\overline{\frac{a^4\dot{x}^2+4a^2x^2\dot{x}^2+4x^4\dot{x}^2}{a^4}}\right|^{\frac{1}{2}}=$

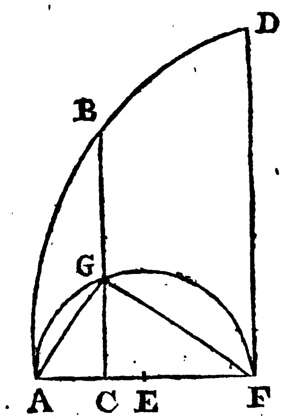
$\frac{a^2\dot{x}+2x^2\dot{x}}{a^2}=$ the *Fluxion* of the Curve GB;

whose *Fluent* (*Art.* 110.) is $z = \frac{a^2x + \frac{2}{3}x^3}{a^2} =$

$x + \frac{2x^3}{3a^2} =$ the Length of the Curve GB required.

EXAMPLE II.

118. *To find the Length of the common Cycloid*.*



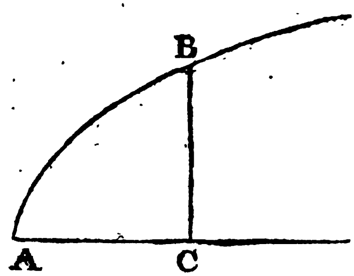
PUT EA or EF, the Radius of the generating Circle, $= a$, Abscifs AC$= x$, Ordinate CB$= y$, CG$= s$, and Arch AB$= z$; then

* See the Generation of this Curve, *Art.* 48. *Note.*

then (as may be found in *Art.* 48.) $\dot{y} = \frac{2a-x}{s}\dot{x}$, and therefore $\dot{y}^2 = \frac{\overline{2a-x}|^2}{s^2}\dot{x}^2$; which substituted for $\dot{y}^2$, makes the general Expression for the *Fluxion* of the Curve (*Art.* 116.) $\dot{z} = \overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}} = \overline{\dot{x}^2 + \frac{\overline{2a-x}|^2}{s^2}\dot{x}^2}|^{\frac{1}{2}}$ $= \overline{\frac{s^2\dot{x}^2+4a^2\dot{x}^2-4ax\dot{x}^2+x^2\dot{x}^2}{s^2}}|^{\frac{1}{2}}$, i.e. (because by the Property of Circles $\overline{GC}|^2 = AC\times CF$, or $s^2=2ax-x^2$,) $\dot{z} = \overline{\frac{4a^2\dot{x}^2-2ax\dot{x}^2}{2ax-x^2}}|^{\frac{1}{2}} = \overline{\frac{2a\dot{x}^2}{x}}|^{\frac{1}{2}} = \overline{2a}|^{\frac{1}{2}}\times x^{-\frac{1}{2}}\dot{x}$; and the *Fluent* of this (by *Art.* 110.) is $z = \overline{2a}|^{\frac{1}{2}}\times 2x^{\frac{1}{2}} = 2\times\overline{2ax}|^{\frac{1}{2}}$ = twice the Chord AG: (for, the Triangles FAG and GAC being similar, FA : AG :: GA : AC, or $\overline{FA\times AC}|^{\frac{1}{2}} = AG$, i. e. $\overline{2ax}|^{\frac{1}{2}} = AG$.) Whence, (by writing $2a$ for x,) the Length of the Semi-Cycloid AD, appears to be equal to twice the Diameter AF of its generating Semi-circle.

EXAMPLE III.

119. *To find the Length of the common Parabola.*



PUT the Parameter $= a$, Abſciſs AC$=x$, Ordinate CB $=y$, and Curve AB$=z$; then (*Art.* 41.) $\dot{x} = \frac{2y\dot{y}}{a}$, and therefore $\dot{x}^2 = \frac{4y^2\dot{y}^2}{a^2}$; which ſubſtituted for $\dot{x}^2$, makes $\dot{z} = \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}}$ (the general Expreſſion for the *Fluxion* of the Length of the Curve, *Art.* 116.) $= \overline{\frac{4y^2\dot{y}^2}{a^2} + \dot{y}^2}|^{\frac{1}{2}} = \frac{\dot{y}}{a} \times \overline{a^2 + 4y^2}|^{\frac{1}{2}}$; which thrown into an infinite Series, is $\frac{\dot{y}}{a} \times : a + \frac{2y^2}{a} - \frac{2y^4}{a^3} + \frac{4y^6}{a^5} -$ &c. i. e. $\dot{z} = \dot{y} + \frac{2y^2\dot{y}}{a^2}$

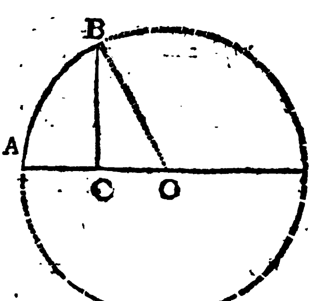
$-\frac{2y^4\dot{y}}{a^4}+\frac{4y^6\dot{y}}{a^6}-$ &c. And the *Fluent* of this Series (*Art.* 110.) is $z=y+\frac{2y^3}{3a^2}-\frac{2y^5}{5a^4}+\frac{4y^7}{7a^6}-$ &c. $=$ the Length of the Curve AB required.

Or thus. The above $\dot{z}=\frac{\dot{y}}{a}\times\overline{a^2+4y^2}|^{\frac{1}{2}}$ is $=\frac{\dot{y}\times\overline{a^2+4y^2}}{a\times\overline{a^2+4y^2}|^{\frac{1}{2}}}=\frac{a^2y\dot{y}+4y^3\dot{y}}{a\times\overline{a^2y^2+4y^4}|^{\frac{1}{2}}}=\frac{\frac{1}{2}a^2y\dot{y}+4y^3\dot{y}}{a\times\overline{a^2y^2+4y^4}|^{\frac{1}{2}}}+\frac{\frac{1}{2}a^2\dot{y}}{a\times\overline{a^2+4y^2}|^{\frac{1}{2}}}=\frac{1}{a}\times\overline{a^2y^2+4y^4}|^{-\frac{1}{2}}\times\overline{\frac{1}{2}a^2y\dot{y}+4y^3\dot{y}}+\frac{1}{4}a\times\frac{\dot{y}}{\overline{\frac{1}{4}a^2+y^2}|^{\frac{1}{2}}}$. Now (by *Art.* 111.) the *Fluent* of the first of these two Terms is $=\frac{1}{2a}\times\overline{a^2y^2+4y^4}|^{\frac{1}{2}}=\frac{y}{a}\times\overline{\frac{1}{4}a^2+y^2}|^{\frac{1}{2}}$; and the *Fluent* of the last of the said two Terms (by *Art.* 114.) is $=\frac{1}{4}a\times$ hyp. Log. of $\overline{y+\overline{\frac{1}{4}a^2+y^2}|^{\frac{1}{2}}}$; therefore $z=\frac{y}{a}\times\overline{\frac{1}{4}a^2+y^2}|^{\frac{1}{2}}+\frac{1}{4}a\times$ hyp. Log. of $\overline{y+\overline{\frac{1}{4}a^2+y^2}|^{\frac{1}{2}}}$: But, since

since when z and y vanish or become $=0$, (as at the Vertex A,) this Fluent becomes $= \frac{1}{2}a \times$ hyp. Log. of $\frac{1}{2}a$; therefore the said *Fluent* being corrected (*Art.* 115. *Ex.* 5.) makes the true Value of z or the Length of the Curve AB $= \frac{y}{a} \times \overline{\frac{1}{4}a^2+y^2}|^{\frac{1}{2}} + \frac{1}{2}a \times$ hyp. Log. of $\overline{y + \overline{\frac{1}{4}a^2+y^2}|^{\frac{1}{2}}} - \frac{1}{2}a \times$ hyp. Log. of $\frac{1}{2}a = \frac{y}{a} \times \overline{\frac{1}{4}a^2+y^2}|^{\frac{1}{2}} + \frac{1}{2}a \times$ hyp. Log. of $\frac{y + \overline{\frac{1}{4}a^2+y^2}|^{\frac{1}{2}}}{\frac{1}{2}a}$, by the Nature of Logarithms.

EXAMPLE IV.

120. *To find the Length of any Arch of a Circle; and of its whole Circumference.*



PUT the Radius OA $=a$, Abscis AC $=x$, Ordinate CB $=y$, Arch AB $=z$: then (*Art.* 40.) $\dot{x} =$

$\dot{x} = \frac{y\dot{y}}{a-x} =$ (because $a-x = CO = \overline{\overline{OB}^2 - \overline{BC}^2}|^{\frac{1}{2}} = \overline{a^2-y^2}|^{\frac{1}{2}}$,) $\frac{y\dot{y}}{\overline{a^2-y^2}|^{\frac{1}{2}}}$; therefore $\dot{x}^2 = \frac{y^2\dot{y}^2}{a^2-y^2}$; which substituted for $\dot{x}^2$, makes $\dot{z} = \overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}$ (the general Expression for the *Fluxion* of the Length of the Curve, *Art.* 116.) $= \overline{\frac{y^2\dot{y}^2}{a^2-y^2}+\dot{y}^2}|^{\frac{1}{2}} = \frac{a\dot{y}}{\overline{a^2-y^2}|^{\frac{1}{2}}} = a\dot{y} \times \overline{a^2-y^2}|^{-\frac{1}{2}}$; which thrown into an infinite Series, is $\dot{z} = a\dot{y} \times : \frac{1}{a} + \frac{y^2}{2a^3} + \frac{3y^4}{8a^5} + \frac{15y^6}{48a^7} + \frac{105y^8}{384a^9} + \frac{945y^{10}}{3840a^{11}} + \frac{10395y^{12}}{46080a^{13}} + \frac{135135y^{14}}{645120a^{15}} +$ &c. $= \dot{y} + \frac{y^2\dot{y}}{2a^2} + \frac{3y^4\dot{y}}{8a^4} + \frac{15y^6\dot{y}}{48a^6} + \frac{105y^8\dot{y}}{384a^8} + \frac{945y^{10}\dot{y}}{3840a^{10}} + \frac{10395y^{12}\dot{y}}{46080a^{12}} + \frac{135135y^{14}\dot{y}}{645120a^{14}} +$ &c. And the *Fluent* of this Series, (*Art.* 110.) is $z = y + \frac{y^3}{6a^2} +$

$\frac{3y^5}{40a^4}$

$$\frac{3y^5}{40a^4}+\frac{15y^7}{336a^6}+\frac{105y^9}{3456a^8}+\frac{945y^{11}}{42240a^{10}}+\frac{10395y^{13}}{599040a^{12}}+\frac{135135y^{15}}{9676800a^{14}}+\text{\&c.}=$$ the Length of the Arch AB.

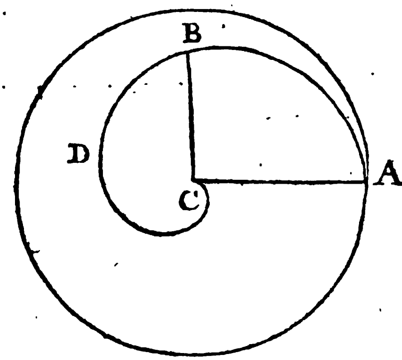
Now, if we suppose the Radius OA $=a=1$, and the $\angle$AOB $=30°$. Then (because the Sine of any Arch is equal to half the Chord of twice that Arch, and the Chord of 60°. is equal to the Radius,) the Sine, or Ordinate CB $=y$, will be $=\frac{1}{2}$; and therefore, the Terms of the above *Fluent* reduced into Decimal Fractions, and placed under one another, will stand thus; *viz.*

.500000000 &c.

	.500000000 &c.		
	.020833333		
	.002343750		
	.000348772		
	.000059339		
	.000010923		
	.000002118		
	.000000426		
	&c.		
the Sum of which is =	.5235987 &c.	=	the Length of the Arch AB:
which multiplied by	 12		
is =	6.283185 &c.	=	the Periphery of the Circle whose *Radius*
is 1: And therefore	3.141592 &c.	=	the Periphery of a Circle whose *Diameter*
is 1.			

Example V.

121. *To find the Length of any Arch of Archimedes's Spiral**.



Put CA the Radius of the generating Circle $= b$, and the Circumference of it $= a$: alſo, put the Ordinate CB$= y$, and the Length of the required Arch CDB $= z$; and a circular Arch, whoſe Radius is CB, $= x$. Then, (as was found in *Art.* 54.) $\dot{x} = \frac{ay\dot{y}}{b^2}$; and therefore $\dot{x}^2 = \frac{a^2y^2\dot{y}^2}{b^4}$; which ſubſtituted for $\dot{x}^2$, makes the general Expreſſion for the *Fluxion* of the Curve, *viz.*

$\dot{z} =$

* See how this Curve is generated, *Art.* 54. *Note.*

$$\dot{z} = \overline{\dot{x}^2 + \dot{y}^2}\big|^{\frac{1}{2}} \ (Art.\ 116.) = \overline{\frac{a^2y^2\dot{y}^2}{b^4} + \dot{y}^2}\Big|^{\frac{1}{2}} =$$

$$\frac{\dot{y}}{b^2} \times \overline{a^2y^2 + b^4}\big|^{\frac{1}{2}} = \frac{\dot{y} \times \overline{a^2y^2 + b^4}}{b^2 \times \overline{a^2y^2 + b^4}\big|^{\frac{1}{2}}} = \frac{a^2y^3\dot{y} + b^4y\dot{y}}{b^2 \times \overline{a^2y^4 + b^4y^2}\big|^{\frac{1}{2}}}$$

$$= \frac{a^2y^3\dot{y} + \frac{1}{2}b^4y\dot{y}}{b^2 \times \overline{a^2y^4 + b^4y^2}\big|^{\frac{1}{2}}} + \frac{\frac{1}{2}b^4\dot{y}}{b^2 \times \overline{a^2y^2 + b^4}\big|^{\frac{1}{2}}} = \frac{1}{b^2} \times$$

$$\overline{a^2y^4 + b^4y^2}\big|^{-\frac{1}{2}} \times \overline{a^2y^3\dot{y} + \tfrac{1}{2}b^4y\dot{y}} + \frac{b^2}{2a} \times \frac{a\dot{y}}{\overline{a^2y^2 + b^4}\big|^{\frac{1}{2}}}.$$

Now, the *Fluent* of the first of these two Terms is found by *Art.* 111. $= \frac{1}{2b^2} \times \overline{a^2y^4 + b^4y^2}\big|^{\frac{1}{2}}$; and the *Fluent* of the second is found by *Art.* 114. $= \frac{b^2}{2a} \times$ hyp. Log. of $\overline{ay + \overline{a^2y^2 + b^4}}$: therefore $z = \frac{1}{2b^2} \times \overline{a^2y^4 + b^4y^2}\big|^{\frac{1}{2}} + \frac{b^2}{2a} \times$ hyp. Log. of $\overline{ay + \overline{a^2y^2 + b^4}\big|^{\frac{1}{2}}}$: but when z and y are $= 0$, this Equation becomes $0 = \frac{b^2}{2a} \times$ hyp. Log. of b^2; and therefore, the *Fluent* corrected,

(*Art.* 115.) makes the true Value of $z =$

$\frac{1}{2b^2} \times \overline{a^2y^4+b^4y^2}|^{\frac{1}{2}} + \frac{b^2}{2a} \times$ hyp. Log. of

$\overline{ay+\overline{a^2y^2+b^4}|^{\frac{1}{2}}} - \frac{b^2}{2a} \times$ hyp. Log. of $b^2 =$ (by

the Nature of Logarithms; the Difference of the Logarithms of any two Numbers being equal to the Logarithm of their Quotient;)

$\frac{1}{2b^2} \times \overline{a^2y^4+b^4y^2}|^{\frac{1}{2}} + \frac{b^2}{2a} \times$ hyp. Log. of

$\frac{ay+\overline{a^2y^2+b^4}|^{\frac{1}{2}}}{b^2} =$ the Length of the Arch

CDB required: And therefore, by ſubſtituting the Radius b for the Ordinate y, we ſhall have the Length of the whole Spiral CDBA

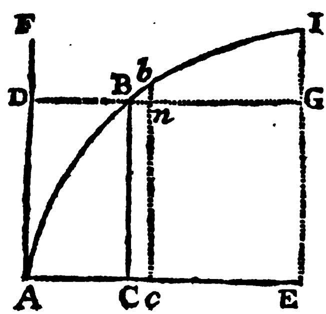
$= \frac{1}{2} \times \overline{a^2+b^2}|^{\frac{1}{2}} + \frac{b^2}{2a} \times$ hyp. Log. of

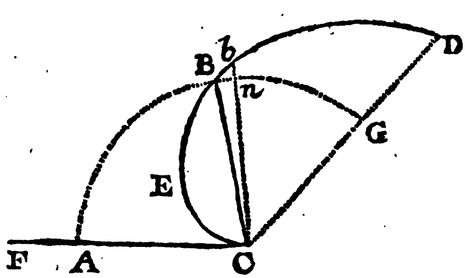
$\frac{a+\overline{a^2+b^2}|^{\frac{1}{2}}}{b}$.

CHAP.

CHAP. IV.

Of finding the Areas of Curve-lined Spaces.





122. IN Curves whose Ordinates are referred to an Axis, (*Fig.* 1.) let *bc* be conceived indefinitely near and parallel to the perpendicular Ordinate BC, and B*n* equal and parallel to C*c* the Increment of the Abscifs AC: Then, because *bn* bears no assignable Ratio to BC, *bc* may be taken as equal to BC or *nc*; and the Trapezium BC*cb* as

as equal to the Parallelogram BC*cn*: But, BC*cb* is the *Moment* or *Increment* of the Curvilineal Space ABC; *i. e.* (if we put AC$=x$, CB$=y$, and C*c*$=x'$,) the *Moment* or *Increment* of the Space ABC is $=yx'$; and therefore (*Art.* 6.) the *Fluxion* of it is $=y\dot{x}$.——In like Manner, in *Spirals*, or those Curves whose Ordinates are referred to a fixt or central Point, (*Fig.* 2.) let *b*C be conceived indefinitely near to the Ordinate BC, and the little circular Arch B*n* (whose Radius is CB,) be supposed a little right Line perpendicular to BC: then, *bn* having less than any assignable Ratio to *n*C, BC*n* may be considered as equal to BC*b* the *Moment* or *Increment* of the Curvilineal Space BECB; *i. e.* (if we put CB$=y$, and B*n* $=x'$,) the *Moment* or *Increment* of the Space CEBC is $=\frac{1}{2}yx'$, or its *Fluxion* (that is, the *Velocity* with which it flows) $=\frac{1}{2}y\dot{x}$.

123. *Or*, without introducing *Increments* or indefinitely small Quantities: Let the curvilineal Space AEI and Parallelogram AG, (*Fig.* 1.) be generated by the perpendicular and indefinite right Line AF, moving with a parallel Motion along the Axis AE: Then, it is evident, the curvilineal Space will flow *slower*, or with a *less* Degree of Velocity, than

than the Parallelogram, *before* the ſaid generating Line arrives at the Term CB; and *afterwards*, *faſter*, or with a *greater* Degree of Velocity; and therefore, *at* the ſaid Term, they will flow with *one* and the *ſame* Degree of Velocity; that is, *at* the Term CB, the *Fluxions* of the curvilineal Space and Parallelogram will be *equal:* but, it is plain, the *Fluxion* of the Parallelogram, at the Term CB, is equal to DA or BC drawn into the *Fluxion* of AC; therefore, the *Fluxion* of the curvilineal Space ACB, is equal to the Ordinate BC drawn into the *Fluxion* of the Abſciſs AC; that is, (putting AC$=x$, and CB$=y$,) the *Fluxion* of the curvilineal Space ACB is $=y\dot{x}$.———In *Fig.* 2. let the curvilineal Space CEDC be generated by the variable right Line CF turning round the Center C; and, at the ſame Time, let the Sector CAGC be deſcribed by the Radius CA: then, it is plain, *before* the Line CF comes to be in the Situation C B, the ſpiral Space will increaſe *ſlower*, or flow with a *leſs* Degree of Velocity, than the circular Space or Sector CAG; and *afterwards*, *faſter*, or with a *greater* Degree of Velocity; and therefore, *at* the Term CB, they will increaſe or flow with an *equal* Degree of Velocity: But, it is evi-

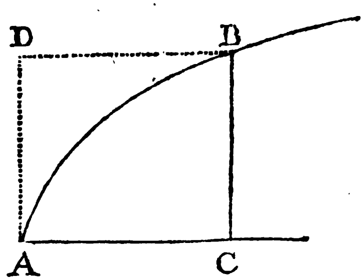
evident, the Velocity with which the Sector enlarges, is equal to half its Radius drawn into the Velocity with which its Arch is described; therefore, the Velocity with which the curvilineal Space CEBC is increased, at the Term CB, is equal to $\frac{1}{2}$ CB drawn into the Velocity of the Point A or B moving along the Arch AG at the Point B; that is, the *Fluxion* of the said curvilineal Space, is equal to $\frac{1}{2}$ CB drawn into the *Fluxion* of the circular Arch AB; or (putting the Ordinate CB$=y$, and Arch AB$=x$,) the *Fluxion* of the curvilineal Space CEBC is $=\frac{1}{2}y\dot{x}$; as before.

124. WHEREFORE, when the Curve is referred to an Axis (*Fig.* 1.) find the Value of y in Terms of x, by Help of the Equation of the given Curve, which multiply by $\dot{x}$; or, find the Value of $\dot{x}$ in Terms of $\dot{y}$, which multiply by y; then, the *Fluent* of the resulting fluxional Expression being found, will give the Area of the curvilineal Space ABC required. And, when the Curve is referred to a fixt or central Point C, (*Fig.* 2.) find the Value of $\dot{x}$ in Terms of $\dot{y}$, from the Properties of the given Curve; then multiply this Value of $\dot{x}$ by $\frac{1}{2}y$, and find the *Fluent*; and

and it will give the Area of the Space required.

EXAMPLE I.

125. *To find the Area of the common parabolic Space* ABCA.



PUT AC$=x$, CB$=y$, and the Parameter $=1$. Then, by the Nature of the Curve, $x=y^2$, or $x^{\frac{1}{2}}=y$; therefore $y\dot{x}$ (the *Fluxion* of the Area, *Art.* 123.) $=x^{\frac{1}{2}}\dot{x}$; the *Fluent* of which, (*Art.* 110.) is $=\frac{2}{3}x^{\frac{3}{2}}=$ (by substituting y for its Value $x^{\frac{1}{2}}$,) $\frac{2}{3}xy=$ the Area ABC required, So that the Area of every common Parabolical Space ABCA, is always equal to two-third-Parts of its circumscribing Parallelogram ADBC.

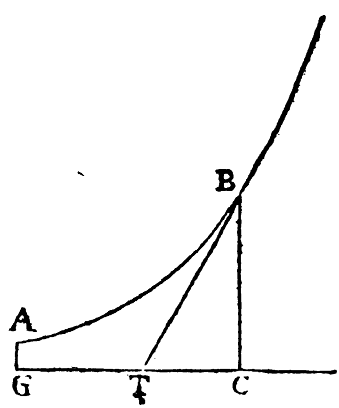
Or thus. The *Fluxion* of the above Equation of the Curve, *viz.* of $x=y^2$, is $\dot{x}=2y\dot{y}$;

 which

which multiplied by y, makes the *Fluxion* of the Area ABC $= y\dot{x}$ (*Art.* 123.) $= 2y^2\dot{y}$; the *Fluent* of which (*Art.* 110.) is $= \frac{2}{3}y^3 =$ (by writing x for y^2,) $\frac{2}{3}xy$; as before.

EXAMPLE II.

126. *To find the Area of the Space* ABCG; *the Property of the Curve* AB *being such, that its Subtangent* CT *is invariable, or always of the same Value.* (See *Art.* 78.)

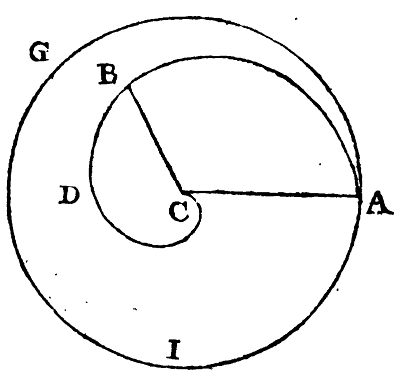


PUT the given Subtangent CT $= a$, GA $= b$, GC $= x$, and CB $= y$. Then, (See *Art.* 37.) $a = \frac{\dot{x}y}{\dot{y}}$; therefore $\dot{x} = \frac{a\dot{y}}{y}$;

$\frac{a\dot{y}}{y}$; which multiplied by y, makes the *Fluxion* of the Space, *viz.* $y\dot{x}$(*Art.* 123.)$=a\dot{y}$; and the *Fluent* of this (*Art.* 108.) is $=ay$: but, when the Area of the Space is $=0$, or $y=b$, this Expression for the *Fluent* becomes $=ab$; and therefore, (*Art.* 115.) the *Fluent* corrected, is $ay-ab=$the Area of the Space ABCG required.

EXAMPLE III.

127. *To find the Area of the Space* CDBC; *the Curve* CDBA *being the Spiral of* Archimedes*.



PUT the Circumference of the generating Circle AIGA $=a$, and its Radius CA$=b$;

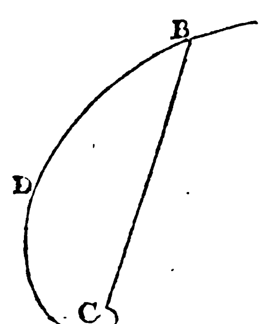
 also,

* See the Generation of this Curve, *Art.* 54. *Note.*

alſo, put the Ordinate CB$=y$. Now, (as was found in *Art.* 54.) $\dot{x}=\frac{ay\dot{y}}{b^2}$; which therefore being multiplied by $\frac{1}{2}y$, makes the *Fluxion* of the Space, *viz.* $\frac{1}{2}y\dot{x}$ (*Art.* 123.) $= \frac{ay\dot{y}}{b^2}\times\frac{1}{2}y=\frac{ay^2\dot{y}}{2b^2}$; the *Fluent* of which, (*Art.* 110.) is $=\frac{ay^3}{6b^2}=$ the Area of the Space CDBC required: And therefore, by ſubſtituting b for y, we have the Area of the whole ſpiral Space CDBAC $=\frac{1}{6}ab$; which, becauſe the Area of a Circle is equal to its Periphery drawn into half the Radius, is $=\frac{1}{3}$ of the Area of the generating Circle.

EXAMPLE IV.

128. *To find the Area of the Space* CDBC; *the Curve* CDB *being the Logarithmic Spiral.* (See *Art.* 55. and 83.)

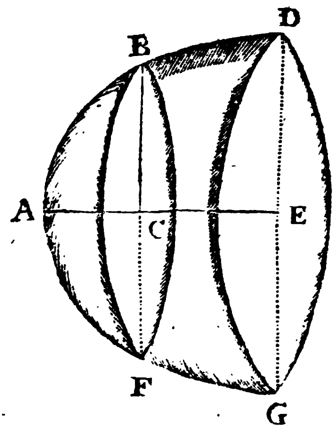


PUT the Ordinate CB $=y$, and the Length of the Curve CDB $=z$; and let two given Quantities, a and b, be to each other in the Ratio of y to z. Then, (as was found in *Art.* 55.) $\dot{y} = \frac{a\dot{x}}{\sqrt{b^2 - a^2}}$; therefore $\dot{x} = \frac{\sqrt{b^2 - a^2}}{a}\dot{y}$, which multiplied by $\frac{1}{2}y$, makes $\frac{1}{2}y\dot{x}$ (the general Expression for the *Fluxion* of the Area, *Art.* 123.) $= \frac{\sqrt{b^2 - a^2}}{2a}y\dot{y}$; the *Fluent* of which (*Art.* 110.) is $= \frac{\sqrt{b^2 - a^2}}{4a} \times y^2 =$ the Area of the Space CDBC required.

CHAP.

CHAP. V.

Of finding the convex Superficies of Solids.



129. LET the Solid ADG be conceived to be generated by an enlarging or variable Circle, (whose increasing Radius is the variable Ordinate of the Curve AD,) moving with a parallel Motion along the Axis from A to E: then will the Velocity with which its convex Superficies flows, be equal to the Periphery of the generating Circle drawn into the Velocity with which it moves along the Curve AD; that is, the *Fluxion* of the said Superficies, at any Term FB, will

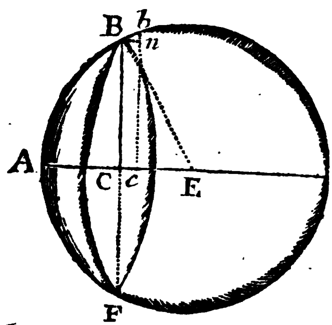
be equal to the Periphery of a Circle, whose Radius is the Ordinate CB, drawn into the *Fluxion* of the Curve at the Point B.——This follows from *Art.* 122. by considering the convex Superficies as always equal to the Area of a plane curvilineal *Fig.* whose Abscissa is equal to the Curve AF, and Ordinate equal to the Circumference of the generating Circle BF, whose Radius is CB.

130. HENCE, if we put the Abscissa AC $= x$, Ordinate CB$=y$, Curve AB$=z$, and $c = 2 \times 3.14159$ &c.$=$the Circumference of a Circle whose Radius is 1; then, because $cy =$ the Circumference of a Circle whose Radius is the Ordinate CB, and (*Art.* 116.) $\dot{z} = \overline{\dot{x}^2 + \dot{y}^2}\big|^{\frac{1}{2}}$; the general Expression for the *Fluxion* of the convex Superficies of any Solid ABF will be $= cy\dot{z}$, or $cy \times \overline{\dot{x}^2 + \dot{y}^2}\big|^{\frac{1}{2}}$; out of which, by Help of the Equation of the given Curve AB, $\dot{x}^2$ or $\dot{y}^2$, &c. may be exterminated; and then, the *Fluent* of the resulting Expression being found, will give the Area of the convex Superficies required: As in the following Examples.

EXAM-

Example I.

131. *To find the Superficies of a Sphere; or the convex Superficies of any Segment of it.*



Put Radius EA or EB$=a$, AC$=x$, CB $=y$, and AB$=z$. Let Bn=Cc express the very first *Moment* of the Increase of AC; nb, of CB; and Bb, of AB; *i. e.* let B$n=x'$, $nb=y'$, and B$b=z'$; then, Bb being considered as a little right Line coinciding with a Tangent to the Point B, the Triangles ECB and bnB will be alike: (for $\angle$ CBn = $\angle$EBb=a right Angle; therefore, $\angle$ EBn being common, $\angle$ CBE = $\angle$ nBb; and, the Angles at C and n being right, the Angles CEB and Bbn must be likewise equal; *ergo*,

ergo, &c.) Wherefore, EB : BC : : bB : Bn, *i. e.* $a : y :: z' : x'$; $\because z' = \frac{ax'}{y}$, or (*Art.* 6.) $\dot{z} = \frac{a\dot{x}}{y}$; which ſubſtituted for $\dot{z}$, makes the general Expreſſion for the *Fluxion* of the convex Superficies, *viz.* $cy\dot{z}$ (*Art.* 130.) $= cy \times \frac{a\dot{x}}{y} = ca\dot{x}$; the *Fluent* of which is $= cax =$ the convex Superficies of the Segment ABF: And, if for x be ſubſtituted $2a$, we ſhall have $2ca^2 =$ the Superficies of the whole Sphere.

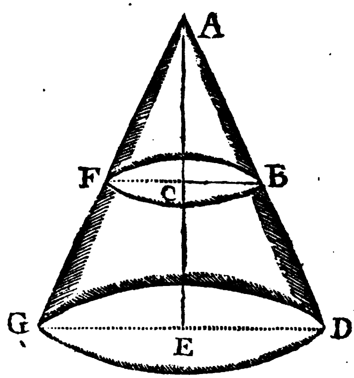
COROLLARIES.

1. THE convex Superficies of any Segment of a Shere, is equal to the Periphery of a great Circle of that Sphere multiplied into the Altitude of the Segment.

2. THE whole Surface or Superficies of any Sphere, is equal to the Periphery of its greateſt Circle multiplied into its Diameter.

EXAMPLE II.

132. *To find the convex Superficies of the right Cone* ADG, *whose Altitude* AE *is given* = a, *and Base-diameter* DG = b.



PUT AC$=x$, and CB$=y$. By Sim. $\triangle s$, AE : ED : : AC : CB; *i.e.* $a : \frac{1}{2}b :: x : y$; $\therefore x = \frac{2ay}{b}$; the *Fluxion* of which Equation is $\dot{x} = \frac{2a\dot{y}}{b}$; $\therefore \dot{x}^2 = \frac{4a^2\dot{y}^2}{b^2}$; which substituted for $\dot{x}^2$, makes the general Expression for the *Fluxion* of the convex Superficies, *viz.*

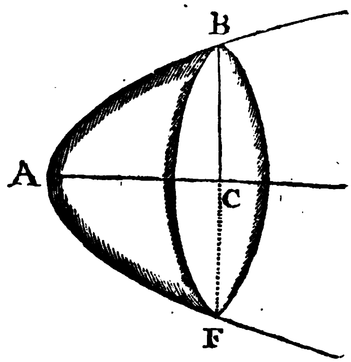
viz. $c\dot{y} \times \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}}$ (*Art.* 130.) $= c y \times$ $\overline{\frac{4a^2\dot{y}^2}{b^2} + \dot{y}^2}|^{\frac{1}{2}} = \frac{c}{b} \times \overline{4a^2 + b^2}|^{\frac{1}{2}} y\dot{y}$; the *Fluent* of which is $= \frac{c}{2b} \times \overline{4a^2 + b^2}|^{\frac{1}{2}} y^2 = \frac{cy^2}{b}$ $\times \overline{a^2 + \frac{1}{4}b^2}|^{\frac{1}{2}} =$ (becauſe $AD = \overline{AE|^2 + ED|^2}|^{\frac{1}{2}}$ $= \overline{a^2 + \frac{1}{4}b^2}|^{\frac{1}{2}}$,) $\frac{cy^2}{b} \times AD =$ the convex Superficies of the Cone ABF: And by writing $\frac{1}{2}b$ for y, we have $\frac{1}{4}cb \times DA =$ the convex Superficies of the Cone ADG.

COROLLARY.

THE convex Superficies of any right Cone, is equal to half the Circumference of its Baſe multiplied into its ſlant Height.

EXAMPLE III.

133. *To find the convex Superficies of the Parabolic Conoid* ABF.



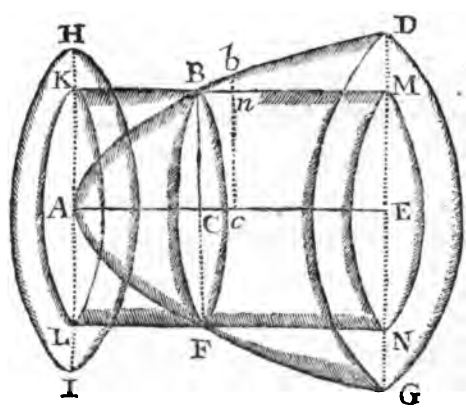
Put the Parameter of the Parabola $= a$, $AC = x$, and $CB = y$. Then, by the Nature of the Curve, $ax = y^2$; the *Fluxion* of which Equation is $a\dot{x} = 2y\dot{y}$; therefore, $\dot{x} = \frac{2y\dot{y}}{a}$, and $\dot{x}^2 = \frac{4y^2\dot{y}^2}{a^2}$; which ſubſtituted for $\dot{x}^2$, makes the general Expreſſion for the *Fluxion* of the convex Superficies, *viz.* $cy \times \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}}$ (*Art.* 130.) $= cy \times \overline{\frac{4y^2\dot{y}^2}{a^2} + \dot{y}^2}|^{\frac{1}{2}} = \frac{c}{a} \times \overline{4y^2 + a^2}|^{\frac{1}{2}} y\dot{y}$; the *Fluent* of which, (*Art.* 111.) is $= \frac{c}{12a} \times \overline{4y^2 + a^2}|^{\frac{3}{2}}$: But, at the Vertex A, where y vaniſhes, or where

$y = 0$,

$y=0$, this Expression becomes $= \frac{c}{12a} \times \overline{a^2}|^{\frac{3}{2}}$ $= \frac{ca^2}{12}$; therefore, the *Fluent* corrected (*Art.* 115.) is $= \frac{c}{12a} \times \overline{4y^2+a^2}|^{\frac{3}{2}} - \frac{ca^2}{12} =$ the convex Superficies of the Solid ABF required.

CHAP. VI.

Of finding the Contents of Solids.



134. LET bc be conceived indefinitely near and parallel to the variable Ordinate BC; and Bn equal and parallel to Cc the

the *Increment* of the variable Abſciſs AC. Now, becauſe the little Parallelogram BC*cn* is expreſſive of the *Increment* or *Moment* of the plane Fig. ABC, (*Art.* 122.) therefore, if the Solid ADG be conceived to be generated by the Revolution of the curvilineal plane Fig. AED round the Axis AE, the indefinitely little Cylinder generated by the ſaid little Parallelogram will expreſs the *Moment* or *Increment* of the Solid at the Term BF; and, becauſe this *Moment* or *Increment* is equal to the Area of the Circle deſcribed by the Ordinate CB drawn into the *Increment* of the Abſciſs AC, therefore (*Art.* 6.) the *Fluxion* of the Solid, at the Term BF, is equal to the Area of a Circle, whoſe Radius is the Ordinate CB, drawn into the *Fluxion* of the Abſciſs AC.

135. *Or*, (without the Suppoſition of any indefinitely ſmall Quantity.)—Let the Cylinder LM be generated by the parallel Motion of the Circle LK along the Line LN; and, at the ſame Time, the Solid ADG, by that of the concentric and variable Circle IH, which, at the Vertex A is ſuppoſed indefinitely ſmall, and continually enlarges as it moves along the Curve AD. Then, it is plain, the Solid ADG will increaſe *ſlower*,

or

or flow with a *less* Degree of Velocity, than the Cylinder LM, *before* the generating Circles arrive at the Term FB; and *afterwards*, *faster*, or with a *greater* Degree of Velocity: therefore, *at* the said Term, (where the two generating Circles become equal, or their Peripheries coincide with each other,) they will increase, or flow, with the *same* or an *equal* Degree of Velocity. But, it is evident, the Velocity with which the Cylinder flows, is equal to the Area of its generating Circle drawn into the Velocity with which it moves along the Line LN; that is, the *Fluxion* of the Cylinder, at the Term FB, is equal to the Area of a Circle whose Radius is CB, drawn into the *Fluxion* of the Line LF or AC: Therefore, the *Fluxion* of the Solid ADG, at the Term BF, is equal to the Area of a Circle, whose Radius is the Ordinate CB, drawn into the *Fluxion* of the Abscis AC; as before.

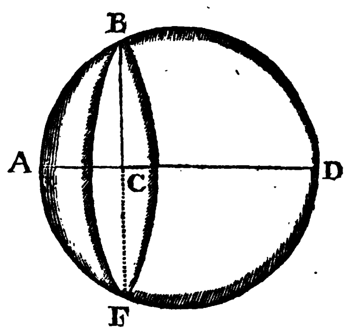
136. HENCE, if we put the Abscis AC $= x$, Ordinate CB $= y$, and $c = 3.14159$ &c. $=$ the Semi-circumference of a Circle whose Radius is 1; then, the general Expression for the *Fluxion* of the solid Content, will be $= cy^2\dot{x}$: out of which, by Help of the Equation of the given Curve, $\dot{x}$ or y^2 may be

ex-

exterminated: and then, the *Fluent* of the resulting *fluxional* Expression being found, will give the Content of the Solid ABF required.

EXAMPLE I.

137. *To find the Content of a Sphere, or of any Segment of it.*



PUT AC$=x$, CB$=y$, and the Diameter AD$=a$; then, by the Property of Circles, $ax-x^2=y^2$; therefore, by writing $ax-x^2$ for y^2, the general Expression for the *Fluxion* of the solid Content, *viz.* $cy^2\dot{x}$ (*Art.* 136.) becomes $= c\dot{x}\times\overline{ax-x^2} = cax\dot{x} - cx^2\dot{x}$;

the *Fluent* of which is $= \frac{cax^2}{2} - \frac{cx^3}{3} =$

$\frac{3cax^2-2cx^3}{6}$ = the Content of the Segment ABF:

ABF: And, if a be ſubſtituted for x, we ſhall have $\frac{3ca^3-2ca^3}{6}=\frac{1}{6}ca^3=$ the Content of the whole Sphere ABDF. Hence, becauſe four Times the Area of a great Circle of the Sphere is $=ca^2$, and the Content of a Cylinder circumſcribing the Sphere is $=\frac{1}{4}ca^3$, we have the following

COROLLARY.

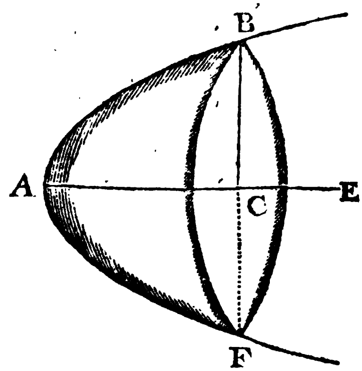
THE Content of any Sphere, is equal to four Times the Area of its greateſt Circle multiplied into $\frac{1}{6}$th Part of its Axis; or, equal to two-third-Parts of its circumſcribing Cylinder.

EXAMPLE II.

138. *To find the Content of the Parabolic Conoid* ABF, *generated by the conic or common Semi-Parabolic Space* ABC *revolving round the Axis* AB.

PUT the Parameter $=a$, AC $=x$, and CB $=y$; then, by the Nature of the Curve, $ax=y^2$. Now, by ſubſtituting ax for y^2, we ſhall have the general Expreſſion for the

 Fluxion



Fluxion of the ſolid Content, *viz.* $cy^2\dot{x}$ (*Art.* 136.) $= cax\dot{x}$; the *Fluent* of which is $= \frac{1}{2}cax^2 =$ (by writing y^2 for ax,) $\frac{1}{2}cxy^2 =$ the Content of the Parabolic Conoid ABF required.

Or thus. The *Fluxion* of the Equation of the Curve, *viz.* of $ax = y^2$, is $a\dot{x} = 2y\dot{y}$; therefore $\dot{x} = \frac{2y\dot{y}}{a}$; which ſubſtituted for $\dot{x}$, makes the general Expreſſion for the *Fluxion* of the ſolid Content, *viz.* $cy^2\dot{x}$ (*Art.* 136.) $= cy^2 \times \frac{2y\dot{y}}{a} = \frac{2cy^3\dot{y}}{a}$; the *Fluent* of which is $= \frac{cy^4}{2a} =$ (by writing ax for y^2,) $\frac{1}{2}cxy^2 =$ the ſolid Content of the Conoid; as before.

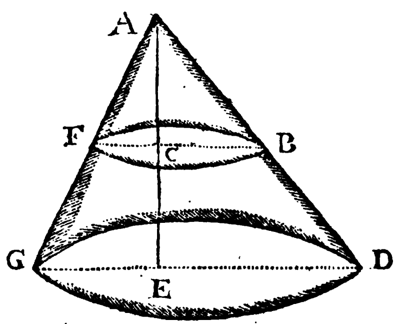
Corol-

COROLLARY.

THE Content of every common Parabolic Conoid, is equal $\frac{1}{2}$ of its circumscribing Cylinder.

EXAMPLE III.

139. *To find the Content of any Cone* ADG *whose Base is a Circle.*



PUT the given Altitude AE $=a$; and Base-diameter GD$=b$. Let FB be parallel to GD; and put AC$=x$, and $c=.78539$ &c. Then, by Sim. $\triangle s$, AE : GD : : AC : FB, *i.e.* $a:b::x:\frac{bx}{a}=$ FB; therfore, the Area of the Circle FB is $=\overline{\frac{bx}{a}}\Big|^2\times c=\frac{cb^2x^2}{a^2}$;

 which

which drawn into $\dot{x}$, is $\frac{cb^2x^2\dot{x}}{a^2}$ = the *Fluxion* of the Content of the Cone at the Term FB, (*Art.* 135.) the *Fluent* of which is $\frac{cb^2x^3}{3a^2}$ = the Content of the Cone AFB: And by ſubſtituting a for x, we have $\frac{cb^2a^3}{3a^2} = \frac{1}{3}cb^2a$ = the Content of the Cone ADG required.

COROLLARY.

THE ſolid Content of any Cone whoſe Baſe is a Circle, is equal to the Area of its Baſe multiplied into $\frac{1}{3}$ of its perpendicular Altitude.

140. *N.B.* IF inſtead of the Diameter, we put the Area of the Baſe $= b$; then (becauſe ſimilar plane Figures are as the Squares of their homologous Sides,) we ſhall have $a^2 : b :: x^2 : \frac{bx^2}{a^2}$ = the Area of the Section FB, which drawn into $\dot{x}$ is $= \frac{bx^2\dot{x}}{a^2}$ = the *Fluxion* of the Cone AFB; the *Fluent* of which, gives the Content of the ſaid Cone $= \frac{1}{3} \times \frac{bx^3}{a^2}$: And by

by writing a for x, we have the Content of the whole Cone AGD $= \frac{1}{3}ab$. And, as b may here ſtand for the Area of the Baſe of any Pyramid, it follows, that, the ſolid Content of any Pyramid whatever, is $=$ the Area of its Baſe multiplied into $\frac{1}{3}$ of its perpendicular Altitude.

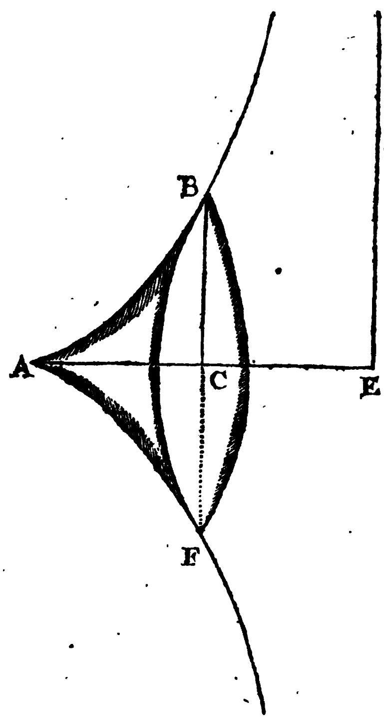
EXAMPLE IV.

141. *To find the Content of the Solid* ABF, *generated by the Ciſſoidal Space* ABC *revolving round the Axis* AE *.

PUT the Axis AE$=a$, Abſciſs AC$=x$, and Ordinate CB$=y$; then, (as was found in *Art.* 47.) $x^3=ay^2-xy^2$: Therefore $y^2 = \frac{x^3}{a-x}$; which ſubſtituted for y^2, makes the general Expreſſion for the *Fluxion* of the ſolid Content, *viz.* $cy^2\dot{x}$ (*Art.* 136.) $= \frac{cx^3\dot{x}}{a-x} = \frac{-cx^3\dot{x}}{x-a} = -cx^2\dot{x}-cax\dot{x}-ca^2\dot{x}-\frac{ca^3\dot{x}}{x-a} = -cx^2\dot{x}-cax\dot{x}-ca^2\dot{x}-ca^3 \times \frac{-\dot{x}}{a-x}$; the *Flu-*

ent

* See how the *Ciſſoid* is generated, *Art.* 47. *Note.*



ent of which, (becauſe the *Fluxion* of the hyp. Log. of $\overline{a-x}$ is $\frac{-\dot{x}}{a-x}$, *Art.* 17. or 114.) is $= -\frac{cx^3}{3} - \frac{cax^2}{2} - ca^2x - ca^3 \times$ hyp. Log. of $\overline{a-x}$: But when $x=0$, (as at A,) this *Fluent* becomes $= -ca^3 \times$ hyp. Log. of a: Therefore, the *Fluent corrected*

(by

(by *Art.* 115.) is $= -\frac{cx^3}{3} - \frac{cax^2}{2} - ca^2x$ $- ca^3 \times$ hyp. Log. of $\overline{a - x} + ca^3 \times$ hyp. Log. of $a =$ (becauſe the Difference of the Logarithms of any two Numbers is equal to the Logarithm of their Quotient,) $\frac{2x^2+3ax+6a^2}{6}$ $\times - cx + ca^3 \times$ hyp. Log. of $\frac{a}{a-x} =$ the Content of the Solid ABF required.

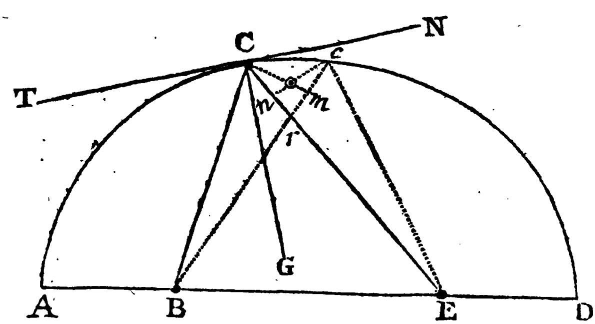
APPENDIX.

APPENDIX.

Miſcellaneous *Queſtions* with their *Anſwers.*

I.

In an Ellipſis ACD, *whoſe Foci are the Points* B *and* E, *if a right Line* CG *be drawn biſſecting the Angle* BCE; *then will the ſaid Line be perpendicular to the Tangent* TCN. *Quære the Demonſtration?*



LET the Point *c* be ſuppoſed indefinitely near to C; and with the Lines BC and E*c*,

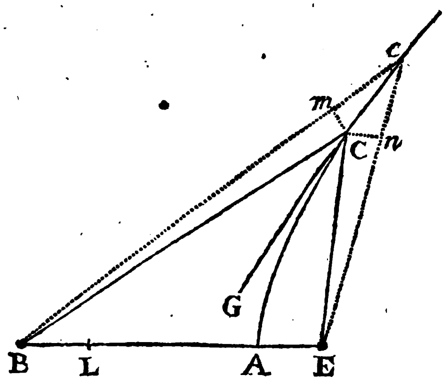
as Radii, describe the Arches C*m* and *cn*; then, if we consider the said Arches as little right Lines perpendicular to B*c* and EC respectively, (because by the Nature of the Curve the Sum of the Lines BC and EC is always the same invariable Quantity, and therefore the Increment *mc* = the Decrement *n*C;) the right-angled Triangles C*n*⊙ and *cm*⊙ will be equal and similar, as will therefore the right-angled Triangles C*mr* and *cnr*; and therefore, if C*c*, the Increment of the Curve AC, be suppos'd to coincide with the Tangent CN, the Angles *rc*C and *r*C*c* will be equal, *i.e.* ∠BCT=∠ECN. Therefore, the right Line CG, which bissects the ∠BCE, makes the ∠GCT=∠GCN= a right Angle. *Q. E. D.*

II.

In an Hyperbola AC, *whose Focus is the Point* E, *and Transverse Diameter is* LA=BA—AE; *the right Line* GC, *which bissects the Angle* BCE, *is a Tangent to the Curve at the Point* C. *Quære the Demonstration?*

SUPPOSE the Point *c* indefinitely near to C; and describe the little Arches C*m* and C*n* with the Radii BC and EC. Now, because

2 by

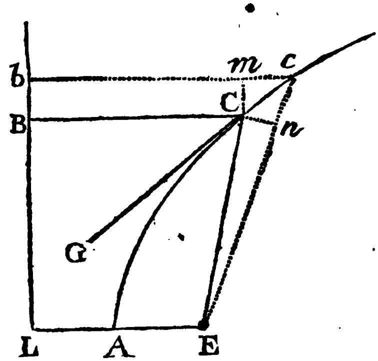


by the Nature of the Curve the Difference of the Lines BC and EC is always the ſame invariable Quantity, (*viz.* = LA,) the Increments *mc* and *nc* are equal: And therefore, if the Arches C*m* and C*n* be conſider'd as little right Lines perpendicular to B*c* and E*c* reſpectively; and C*c*, the Increment of the Curve AC, as coinciding with the Tangent; the little right-angled Triangles C*mc* and C*nc* will be equal and ſimilar. Therefore, the right Line GC, biſſecting the Angle BCE, is a Tangent to the Curve at the Point C. *Q. E. D.*

III.

In a Parabola AC, *whoſe Focus is the Point* E, *if a right Line* CB *be drawn parallel to the Axis, and the Angle* BCE *be biſſected*

by the right Line CG; *then will this Line be a Tangent to the Curve at the Point* C. *Quære the Demonstration?*

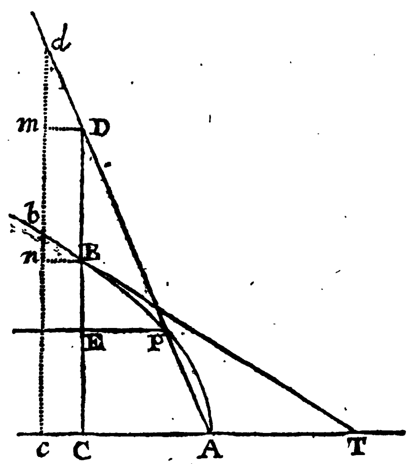


Let the indefinite right Line LB be perpendicular to LE, and LA=AE; draw *bc* indefinitely near and parallel to BC, and C*m* equal and parallel to B*b*; and with EC, as a Radius, describe the little Arch C*n*. Now, because by the Nature of the Curve the Lines BC and EC are always equal to each other, the Increment *mc* = to the Increment *nc*; and therefore, (the indefinitely small Arch C*n* being consider'd as a little right Line perpendicular to E*c*, and the Increment of the Curve, C*c*, as coinciding with the Tangent,) the little right-angled Triangles C*mc* and C*nc* are equal and similar. Consequently, the right Line GC, which bissects the Angle BCE,

BCE, is a Tangent to the Curve at the Point C. *Q. E. D.*

IV.

Quære the Nature of the Curve APB?—*Supposing* CE, *the Distance of the parallel Lines* AC *and* PE, *as also the Angle* CAD, *to be given, and* $\overline{CE}|^{m} \times \overline{CD}|^{n} = \overline{CB}|^{m+n}$.



Put CE $= a$, Abscifs AC $= x$, Ordinate CB $= y$, and CD $= z$. Let cd be conceiv'd indefinitely near and parallel to CD; and Dm, Bn, equal and parallel to Cc; and put $nb = y'$, $md = z'$: And suppose BT a Tangent to the Curve at the Point B. Now, by Sim.

Sim. Δs, DC : CA :: dm : mD, *i. e.* $z : x :: z' :$ $\frac{xz'}{z}$ = Dm or Bn; and bn : nB :: BC : CT, *i. e.* $y' : \frac{xz'}{z} :: y : \frac{xyz'}{zy'}$ = CT; or, $\frac{xy\dot{z}}{z\dot{y}}$ = CT. But, by the *Quest.* $a^m z^n = y^{m+n}$; the *Fluxion* of which Equation is $na^m z^{n-1} \dot{z} = \overline{m+n}\, y^{m+n-1} \dot{y}$; $\because \dot{z} = \frac{\overline{m+n}\, y^{m+n-1} \dot{y}}{na^m z^{n-1}}$; which substituted for $\dot{z}$, makes the above Value of CT (*viz.* $\frac{xy\dot{z}}{z\dot{y}}$) $= \frac{x.\overline{m+n}\, y^{m+n}}{na^m z^n} =$ (because $\frac{y^{m+n}}{a^m z^n} = 1$,) $\frac{m+n}{n} x$.

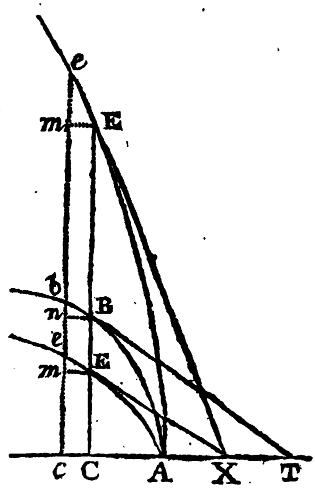
Now, this is an *universal* Expression for the Subtangent of all possible Parabolas: Therefore, when $m = n = 1$, the Subtangent will be $= 2x$; and the Curve, the common or apollonian Parabola. *Q. E. I.*

Note. Tho' this universal Expression for the Subtangent, seems to differ from that in *Art.* 42. yet it will be found to express the very same Thing, if we consider that the Parameter drawn into x, in the Equation of the Curve there, is equal to $a^{\frac{m}{n}} z$ here; for then, m there, will be equal to $\frac{m+n}{n}$ here.

V.

V.

Let TB *be a Tangent to the given Curve* AB: *Then, if another Curve* AE *be so drawn as that its Ordinate* CE *shall always be in a given Ratio to the corresponding Segment of the former Curve; I say,* BT *will be to* TC *as the corresponding Segment of the Curve* AB *is to the Subtangent* CX. *Quære the Demonstration?*



Put $CT = s$, $TB = t$, $CE = y$, $AB = z$: And let ce be suppos'd indefinitely near and parallel to CE; and Em, Bn, equal and parallel to Cc; *i.e.* let $me = \dot{y}$, and $Bb = \dot{z}$.

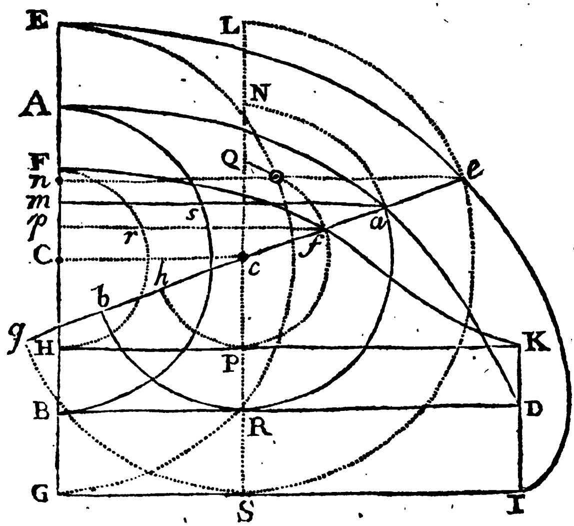
Now,

Now, BT : TC : : bB : Bn, i.e. $t : s : : z' : \frac{sz'}{t}$ = Bn or Em; and em : mE : : EC : CX, *i. e.* $y' : \frac{sz'}{t} : : y : \frac{syz'}{ty'}$ = CX, or $\frac{sy\dot{z}}{t\dot{y}}$ = CX. Let the given Ratio of EC to AB be as a to b, i.e. $y : z : : a : b$, $\because z = \frac{by}{a}$; which Equation put into *Fluxions*, is $\dot{z} = \frac{b\dot{y}}{a}$; and this substituted for $\dot{z}$, makes the above $\frac{sy\dot{z}}{t\dot{y}}$ = CX $= \frac{bsy}{at}$ (which, by writing z for its Value $\frac{by}{a}$, is) $= \frac{sz}{t}$; therefore $t : s : : z$: CX, *i. e.* BT : TC : : AB : CX. *Q. E. D.*

VI.

If a Semicircle AB *be rolled along upon an indefinite right Line* BD *until it measures out a Line equal to its Circumference; then, the Curve* AD, *described by the Point* A, *being the common Semi-cycloid*, (Art. 48.) *I say, the Curve* EI, *described by the Point* E *taken* without *the Circle, will be the* contracted *or* curtate *Semi-cycloid; and the Curve* FK, *de-*

deſribed by the Point F *taken* within *the Circle, will be the* inflected *Semi-cycloid. Quære the Demonſtration?* [*See* Art. 62. *and* Prob. 7 *of this* Appendix.]



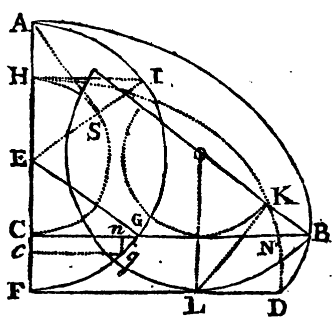
As all the Cycloids, properly ſpeaking, are generated by the Revolution of one and the ſame Circle AB; ſo therefore, their Baſes will be all equal to BD, *viz.* = the Semi-circle AB. Let the Generant move on till it comes in the Poſition *ab*: To the deſcribing Points *a*, *e*, *f*, draw the Ordinates *ma*, *ne*, *pf*; and draw LS equal and parallel to EG. Then, becauſe the correſponding Ordinates in the

Circle are respectively equal, it is evident, that, the Distances from the Extremities of these Ordinates to the corresponding Points in the Curves, will be all equal to the Distance moved over by the common Center C; that is, $sa = oe = rf = Cc$; each of the said Distances therefore is $=$ BR $=$ Arch bR by the Generation. Now, the Sectors Lce and Scg, or Qcf and Pcb, are equal and similar, and similar to the equal Sectors Nca and Rcb: Therefore, $cg : cb :: g\mathrm{S} : b\mathrm{R}$; or, (if we put the Semicircle $eg = a$, Semicircle ab or Base BD $= b$,) as $a : b ::$ Arch gS : Arch bR :: Arch Eo : (BR or) oe; *i. e.* Semicircle EG : GI :: Arch Eo : oe. Wherefore, the Curve EeI is the *curtate* Semi-cycloid.—And, $cb : cb :: b\mathrm{P} : b\mathrm{R}$; or, (if we put the Simicircle $fb = a$, and Semicircle ab or Base BD$= b$,) as $a : b ::$ Arch bP : Arch bR :: Arch Fr : (BR or) rf; *i. e.* Semicirele FH : HK :: Arch Fr : rf. Wherefore, the Curve FfK is the *inflected* Semi-cycloid. *Q. E. D.*

VII.

Let the Curve ABD *be a given* curtate *Semi-cycloid,* i. e. *an* exterior *Semi-cycloid, or a Semi-cycloid the Circumference of whose gene-*

generating Semicircle AGF *is greater than its Base* FD. *Quære the Point of Retrogression* B; *or the Point* C *in the Diameter* AF, *where the Ordinate* CB *is a* Maximum, *or the greatest possible?* [See the preceding *Prob.*]



Put the generating Semicircle AGF$=a$, Base FD$=b$, Radius EG$=c$, EC$=x$, and Arch AG$=z$: And let cg be supposed indefinitely near and parallel to CG, and ng equal and parallel to Cc; *i.e.* let $ng=x'$, and G$g=z'$, Now, by 47E. 1. CG $=\overline{c^2-x^2}|^{\frac{1}{2}}$; and, if the *Increment* Gg be supposed a little right Line perpendicular to the Radius EG, the $\triangle$s CGE and ngG will be similar; and therefore CG : GE : : ng : gG, *i.e.* $\overline{c^2-x^2}|^{\frac{1}{2}}$:

$c :: x' : z' = \dfrac{cx'}{\overline{c^2 - x^2}|^{\frac{1}{2}}}$, or $\dot{z} = \dfrac{c\dot{x}}{\overline{c^2 - x^2}|^{\frac{1}{2}}}$.

By the Nature of Cycloids, $a : b :: z : \dfrac{bz}{a} =$ GB; therefore, CG+GB $= \overline{c^2 - x^2}|^{\frac{1}{2}} + \dfrac{bz}{a}$ = a *Maximum* by the *Quest.* the *Fluxion* of which is $- \dfrac{x\dot{x}}{\overline{c^2 - x^2}|^{\frac{1}{2}}} + \dfrac{b\dot{z}}{a} = 0$; whence $\dot{z} = \dfrac{ax\dot{x}}{b.\ \overline{c^2 - x^2}|^{\frac{1}{2}}} =$ (because $\dot{z}$ by the above is =) $\dfrac{c\dot{x}}{\overline{c^2 - x^2}|^{\frac{1}{2}}}$; which Equation reduced, makes $ax = bc$, or $x = \dfrac{bc}{a} =$ EC. *Q. E. I.*

OBSERVATION.

If the concentric Semicircle HSC be drawn equal to the given Base FD; and the right Line HI be drawn perpendicular to the Diameter AF; then, when the Point I arrives at the Base FD, the Point A will be in the Point of Retrogression B. For,

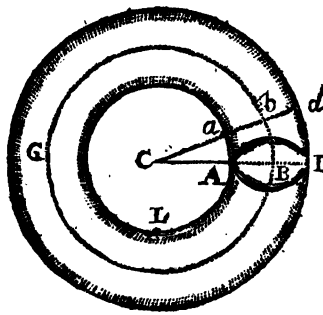
If we draw the Line CN equal and parallel to the Base FD; then, by the last *Prob.* when

when the Point H generates the common Cycloid HKN, the Point A will describe the Curve in *Quest.* And, since by the above Solution, $x = \frac{bc}{a}$, which by Analogy is $a : b :: c : x$, *i.e.* Semicircle AIF : (FD or) Semicircle HSC :: EF : EC; therefore, the Point of Retrogression B, will be in the Base CN, of the common Cycloid, produced: Consequently, if AS be drawn a Tangent to the Semicircle HSC; then, when the said Tangent coincides with the Base CN produced, the Point A will be in the Point of Retrogression B: But, when the Point S comes to the Base CN, the Point I will arrive at, or be in the Base FD. Therefore, &c.

VIII.

Quære the Content of the cylindrical Ring ABDG? — *the Radius* CA *of the inner Circumference being given* $= a$; *and* AD, *the Diameter of the Ring, or of the generating Circle,* $= b$.

Put any Arch LA$=x$, .78539 &c.$=c$: Suppose Cd indefinitely near to CD; and draw the concentric prickt Circle BG, making

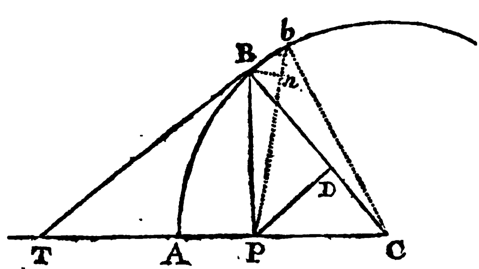


ing AB=BD=$\frac{1}{2}b$: Then, D*d*, B*b*, and A*a*, being considered as little right Lines perpendicular to the Diameter AD, the *Moment* of the Ring, *viz.* A*d*, will be equal to the Area of the Circle AD drawn into the *Increment* B*b*. Now, CA : A*a* : : CB : B*b*, i. e. $a : x' ::$ $a+\frac{1}{2}b : x'+\frac{bx'}{2a} =$ B*b*; and, the Area of the Circle AD $=b^2c$: Therefore, the *Moment* of the Ring is $=b^2c\times\overline{x'+\frac{bx'}{2a}}=b^2cx'+$ $\frac{b^3cx'}{2a}$, or, its *Fluxion* $=b^2c\dot{x}+\frac{b^3c\dot{x}}{2a}$: And the *Fluent* of this, is $=b^2cx+\frac{b^3cx}{2a}=$ $\overline{1+\frac{b}{2a}}\times b^2cx=$ the Content of the Ring from

from L to A; and, by substituting $8ac$ for x, we have the Content of the whole Ring $=$ $\overline{1+\frac{b}{2a}} \times 8ab^2c^2 = b^2c \times \overline{2a+b}.4c =$ the Area of the generating Circle AD drawn into the Circumference of the prickt Circle BG. *Q. E. I.*

IX.

Given the Point P *in the Radius* CA *of the given Circle* AB *&c. Quære the Point* B, *to which a Tangent* TB, *and right Line* PB, *being drawn, the Angle* PBT *may be a* Minimum, *or the least possible?*



Draw the Radius CB. Now, because the Tangent to a Circle is always perpendicular to the Radius, it is evident, that, the Angle PBT will be the least possible when the Angle PBC is the greatest. It is likewise evi-

evident, that, as long as the Angle BPC decreaſes faſter than the Angle BCP increaſes, the Angle PBC will be *increaſing*; and that, when the Angle BCP increaſes faſter than the Angle BPC decreaſes, the ſaid Angle PBC will be *decreaſing*: Therefore, when the Angle PBC is a *Maximum*, the *Fluxions* of the Angles at P and C will be equal; *i. e.* if P*b* be ſuppoſed indefinitely near to PB, and the little circular Arch B*n* be deſcribed with the Radius PB; then, when the Angle PBC is a *Maximum*, the indefinitely ſmall Angles BP*n* and BC*b* will be equal to each other: Therefore, PB : B*n* : : CB : B*b*, or B*b* : B*n* : : CB : PB. Draw PD perpendicular to the Radius CB: Then, the Triangles *bn*B and BDP will be ſimilar; and therefore, *b*B : B*n* : : PB : BD. Hence we have CB : PB : : PB : BD; and, conſequently, the Triangles BDP and BPC are alike; and therefore, the Triangle BPC is right-angled at the Point P; *i. e.* when the Angle PBC is a *Maximum*, or the Angle PBT is a *Minimum*, the right Line PB is perpendicular to the Radius AC. *Q. E. I.*

Or thus. Becauſe, by Trigonometry, CB : Sine ∠BPC : : CP : Sine ∠PBC; therefore, the Angle PBC will be greateſt when the Angle

Angle CPB is a right one; but, when the Angle PBC is the *greateſt*, the Angle PBT is the *leaſt* poſſible: Therefore, when the ſaid Angle PBT is a *Minimum*, the right Line PB will be perpendicular to the Radius AC; as before.

X.

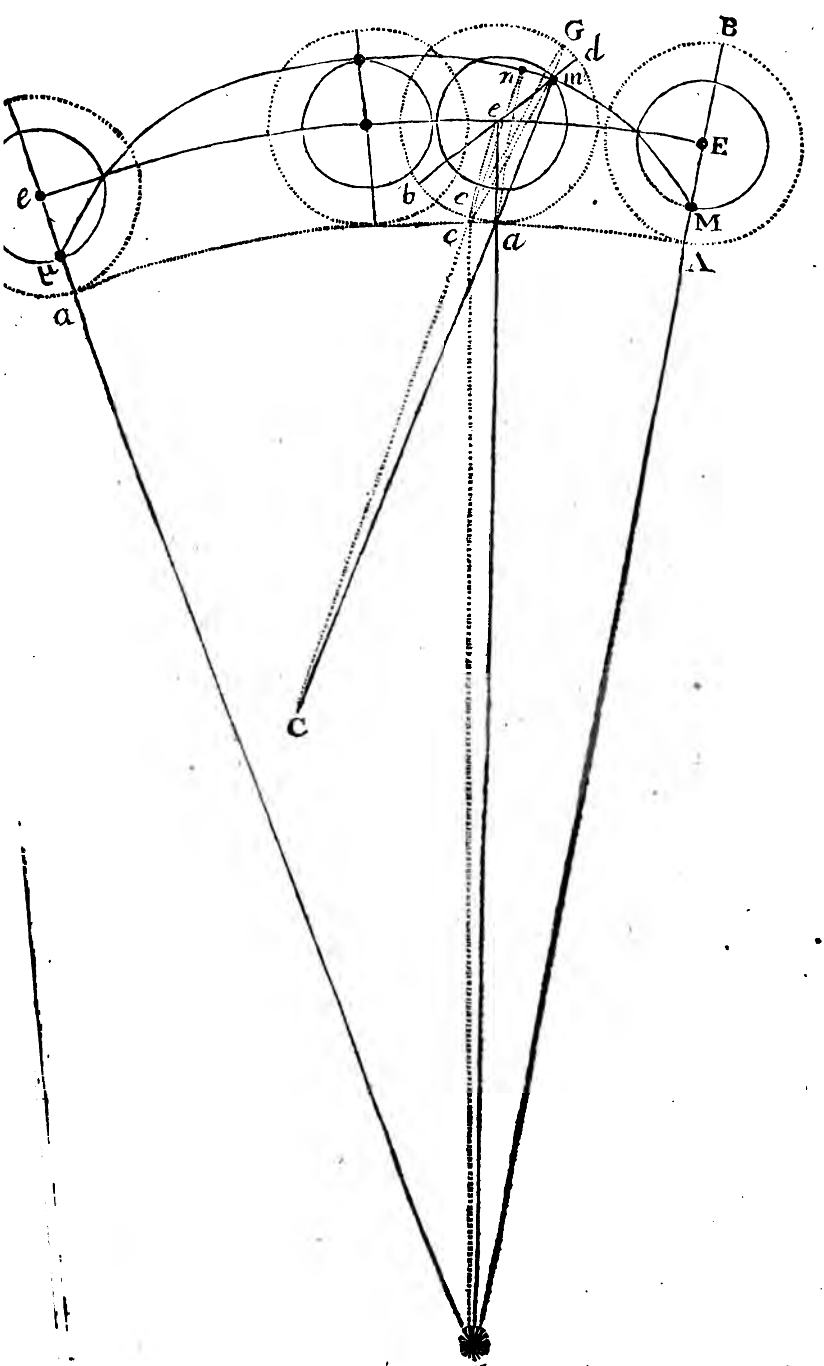
Suppoſe the Earth to revolve in a circular Orbit round the Sun as its Center; and the Moon to revolve round the Earth in the ſame Manner; as alſo, that the Planes of their Orbits do coincide; and that the Diameters of the ſaid Orbits are as 340 *to* 1; *and, laſtly, that the Moon performs* 13.38 *Revolutions to every ſingle Revolution of the Earth. Quære the Nature and Deſcription of the Curve generated by the Center of the Moon? — Or, whether the Curve deſcribed by the Center of the Moon, in one Lunation, be any where* Convex *towards the Sun?*

[See the *Gentleman's Magazine* for *September* 1743, and the *Supplement* for that Year.]

LET S repreſent the Sun; E, the Earth; E*ee*, an Arch of the Orbit of the Earth paſſed over by its Center in one Lunation of the Moon; the Circumference of the Circle

EAB = the concentric Arch A*aa*: Then, (becauſe 13.38—1=12.38 is = the Number of Lunations in a Year, or one Revolution of the Earth,) when the Moon is in Conjunction with the Sun, the Diſtance between the Sun and Moon will be greater than the Diſtance or Radius SA. Now, the Curve deſcribed by the Center of the Moon, is the ſame as that deſcribed by a Point M (EM being the Semidiameter of the Moon's Orbit,) carried round by the Rotation of the Circle EAB on the Arch A*a*: It is therefore of the *Cycloidal* Kind, having a Point of *contrary Flexure*, if *every* Cycloid deſcribed by a Point within the generating Circle, is *inflected* as well upon a *circular* as upon a *rectilinear* Baſe. In Order to determine which,

PUT SA or S*a* = *a*, EA or *ea* = *b*, EM or *em* = *c*, *am* = *r*, and *a*G = *s*; and let *m*C be the Radius of Curvature at any Point *m*, and *n*C indefinitely near it. Then, *ac* and *ac* being the indefinitely ſmall contemporary Arches with *mn*, and conſequently the Triangles *amc* and *anc* equal in all Reſpects; if we conſider the ſaid little Arches *ac* and *ac* as little right Lines perpendicular to the Radii *ec* and S*c*, we ſhall have the $\angle man = \angle cac =$ (becauſe the Angles *eac* and S*ac* added to either


G
d
B
n
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E
e
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a
C
S

ther Side of the Equation make it two right Angles,) $\angle aec + \angle a\mathrm{S}c$. Now, $\mathrm{S}a : ea :: \angle aec : \angle a\mathrm{S}c$; and $\mathrm{S}a : \mathrm{S}a + ae :: \angle aec : \angle aec + \angle a\mathrm{S}c$, i.e. $a : a+b :: \angle aec : \angle man = \frac{a+b}{a} \times \angle aec$. Again; in any Triangle, as Gmc, if the Angles $m\mathrm{G}c$, $mc\mathrm{G}$, and amc the Complement of the obtuse Angle to two right Angles, be indefinitely small, they will be proportional to the opposite Sides mc, $m\mathrm{G}$, and Gc; (which see demonstrated in the *Note* below.) *i. e.* $\mathrm{G}c : m\mathrm{G} :: \angle amc : \angle mc\mathrm{G}$; and $\mathrm{G}c - m\mathrm{G} : \mathrm{G}c :: \angle amc - \angle mc\mathrm{G} : \angle amc$, i. e. $ma : \mathrm{G}a :: \angle a\mathrm{G}c : \angle anc$, or $r : s :: \frac{1}{2}\angle aec : \angle anc = \frac{s}{2r} \angle aec$. And again, $\angle a\mathrm{C}n : \angle an\mathrm{C} :: an : a\mathrm{C}$, *i. e.* $\angle man - \angle anc : \angle anc :: am : a\mathrm{C}$, or $\overline{\frac{a+b}{a} - \frac{s}{2r}} \angle aec : \frac{s}{2r} \angle aec :: r : a\mathrm{C} = \frac{ars}{2ar+2br-as}$. Consequently, $ma + a\mathrm{C} = m\mathrm{C} = \frac{2ar^2+2br^2}{2ar+2br-as} = \frac{r^2}{r - \frac{as}{2a+2b}}$ $=$ the Radius of Curvature at any Point m.

Now,

Now, as this Expreſſion for the Radius of Curvature, muſt be Affirmative on one Side of the Point of Inflection, and Negative on the other; or, ſince r muſt be *more* than $\frac{as}{2a+2b}$ on one Side of the ſaid Point, and on the other *leſs*; therefore, *at* the Point of Inflection, $r = \frac{as}{2a+2b}$; which ſubſtituted for r, makes ($Gm \times ma$, i. e.) $rs - r^2 = \frac{2abs^2 + a^2s^2}{\overline{2a+2b}|^2} =$ (becauſe $Gm \times ma = bm \times md =$) $b^2 - c^2$; from which Equation, we have $s = \frac{\overline{2a+2b}\sqrt{b^2-c^2}}{\sqrt{2ab+a^2}}$. Or, to find r, ſay $2ar + 2br = as$, or $s = \frac{2ar+2br}{a}$; then ($Gm \times ma =$) $rs - r^2 = \frac{ar^2+2br^2}{a} =$ ($bm \times md =$) $b^2 - c^2$; which Equation gives $r = \sqrt{\frac{ab^2-ac^2}{a+2b}}$, when the Point m falls in the Point of *Inflection*. Now, as ma (r) muſt, by the Nature of the Circle, always be *greater* than md; that is, as $\sqrt{\frac{ab^2-ac^2}{a+2b}}$ muſt always be *more* than $b-c$,

and

and consequently, $\frac{ab^2 - ac^2}{a+2b}$ *more* than $\overline{b-c}|^2$, i. e. $\frac{a \times \overline{b+c} \times \overline{b-c}}{a+2b}$ than $\overline{b-c} \times \overline{b-c}$; so therefore, (*em*) c must be always *more* than $\frac{b^2}{a+b}$, in Order to a Point of Inflection's taking Place in the Curve. But, in the present Case, a, b, and c, being as 12.38, 1, and $\frac{13.38}{340}$ or .039; therefore, c is *less* than $\frac{b^2}{a+b}$; and consequently, the Curve M$m\mu$, generated by the Center of the Moon, has *not* a Point of *contrary Flexure*, or, is no where *Convex* towards the Sun. *Q. E. I.*

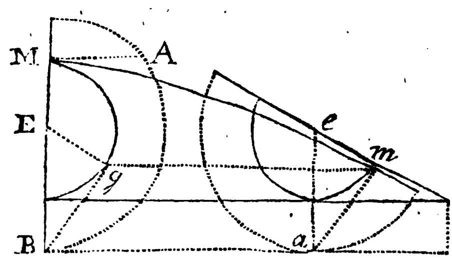
Corollaries.

1. When $r=s$, that is, when the Point m coincides with d, the Radius of Curvature will be $= \frac{2ar^2+2br^2}{ar+2br}$ or $\frac{a+b}{a+2b}2r$; and aC$=$ $\frac{a}{a+2b}r$; by Analogy, $a+2b : a :: r : a$C; that is, SB : SA :: ma : aC.

2. When a is infinite, and $r=s$; that is, when the Base becomes a right Line, and the

the Point m coincides with d, or the Curve is the *common Cycloid*; the Radius of Curvature will be $= 2r$: For then, $2br^2$ and $2br$, will be infinitely little in Comparison of $2ar^2$ and ar, and therefore may be rejected.

3. When a is infinite, that is, when the Base degenerates into a right Line, or the Curve is the common *Inflected* or *Interior Cycloid*, (as in the annexed Fig.) if the Tan-



gent Bg be drawn; the Point of Inflection will be in the Line gm parallel to the Base; that is, if AM be perpendicular to MB; when the Point A comes to the Base, the Point M will be in the Point of Inflection: For, when the Point m is in the Line gm, it is evident, that, am will be equal to Bg, and the Triangles BgE and ame equal in every Respect; and, because the Tangent to a Circle is perpendicular to the Radius, the

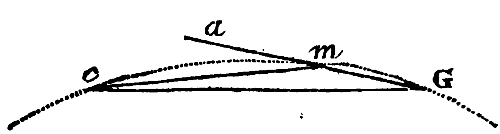
$\angle ame =$ a right Angle: Wherefore, by 47 E. 1. $\overline{ae}|^2 - \overline{em}|^2 = \overline{ma}|^2$; that is, (if m be the Point of Inflection, and not otherwise,)

$$b^2 - c^2 = r^2 = \frac{ab^2 - ac^2}{a+2b}; \text{ i. e. } b^2 - c^2 = \overline{b^2 - c^2}$$

$$\times \frac{a}{a+2b};$$ but, $2b$ being infinitely little in Respect of a, therefore $\frac{a}{a+2b}$ is infinitely near to Equality with $\frac{a}{a}$, i.e. $= 1$; *ergo*, &c.

[*See* Art. 62. *and* Prob. 6th *of this* Appendix.]

Note, THAT the Angles mGc, mcG, and amc, when *indefinitely* small, are as the Sides mc, mG, and Gc, (as they are said to be in the above *Solution*;) may thus be proved.



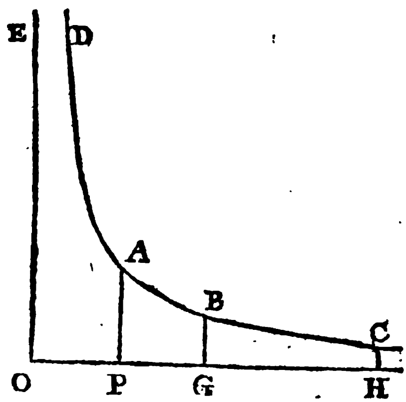
Let the $\triangle$ be circumscribed by the Circle cmG: then will the Arches Gm and mc differ infinitely little from their Chords Gm and mc, and therefore may be taken as equal to them: And, since the Angle at the Center of

of a Circle, is double the Angle at the Periphery; therefore, the Arches or Chords *cm* and *m*G, are the Measures of double the Angles *m*G*c* and *mc*G respectively; or, the Angles *m*G*c* and *mc*G are to each other as their opposite Sides *cm* and *m*G: And, because the Angle *amc* is equal to the Sum of the Angles *m*G*c* and *mc*G; therefore, the Measure of it, is the Sum of the Arches or Chords *cm* and *m*G; which differing infinitely little from the Side or Chord *c*G, may be considered as equal to it: *Ergo*, &c.

XI.

The Fluxion of the Hyperbolic Logarithm of any Quantity, is equal to the Fluxion of that Quantity, divided by the Quantity itself. Quære the Demonstration? (See *Art.* 17.)

Let DABC be an Hyperbola, whose Asymptotes are the perpendicular right Lines OE and OH; and its Parameter AP, which, by a Property of the Curve, is = PO. Now, if AP or PO be put = 1, and GB or HC be drawn parallel to PA; then, the Space PABG will be the hyperbolic Logarithm of $\frac{OG}{OP}$,



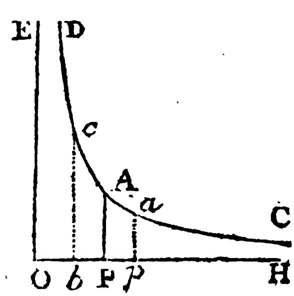
$\frac{OG}{OP}$, *i.e.* of OG; and the Space PACH, the hyperbolic Logarithm of $\frac{OH}{OP}$, *i.e.* of OH; or, the Space GBCH, the hyperbolic Logarithm of $\frac{OH}{OG}$; &c. For, by the Property of the Curve, (See *Stone's De L'Hospital's Conic Sections*, *Art.* 226.) when OP, OG, OH, &c. are in geometrical Proportion continued, the Spaces PABG, and GBCH, &c. will be equal; that is, the Spaces PABG, PACH, &c. will be in arithmetical Progression. So that, the *Fluxion* of the Space PABG, is the *Fluxion* of the hyperbolic Logarithm of OG: Now, the *Fluxion* of this Space, (putting PG $= x$, and GB $= y$,) is (by Art. 122.) $=$

$y\dot{x}$. But, by the Nature of the Curve, OG : OP :: PA : GB, *i.e.* $1+x : 1 :: 1 : y = \frac{1}{1+x}$: Wherefore, $y\dot{x} = \frac{\dot{x}}{1+x} =$ the *Fluxion* of the hyperbolic Logarithm of $1+x$. But, the *Fluxion* of $1+x$, divided by $1+x$, is likewise $= \frac{\dot{x}}{1+x}$; or, the *Fluxion* of the Space PABG expressing the hyp. Log. of the Line OG, is equal to the *Fluxion* of the said Line, divided by the Line itself. *Q. E. D.*

N. B. By a Space *or* Line, *is meant its* Numerical Measure.

XII.

If the Hyperbolic Logarithm of 1 *be* 0, *what is the Hyperbolic Logarithm of* 10? (See Art. 17.)



 LET

Let OE and OH be the Asymptotes of the rectangular Hyperbola DAC, whose Parameter is AP=PO=1. Then, the Area of the Space PA*ap* will be the hyp. Log. of $\frac{Op}{OP}$, *i. e.* of O*p*: The Area of the Space PA*cb* will be the hyp. Log. of $\frac{OP}{Ob}$, *i. e.* of $\frac{1}{Ob}$: And, the Area of the Space *bcap* will be the hyp. Log. of $\frac{Op}{Ob}$. (*See* the preceding *Quest.*) Now, to find these Areas:

1°. Put P*b*=x, and *bc*=y; then, by the Nature of the Curve, $y=\frac{1}{1-x}$; which drawn into $\dot{x}$, is $y\dot{x}=\frac{\dot{x}}{1-x}$ = the Fluxion of P*c*= (by throwing $\frac{\dot{x}}{1-x}$ into a Series) $\dot{x}+x\dot{x}+x^2\dot{x}+x^3\dot{x}+x^4\dot{x}+x^5\dot{x}+x^6\dot{x}+x^7\dot{x}+x^8\dot{x}+x^9\dot{x}+x^{10}\dot{x}+x^{11}\dot{x}+$ &c. Therefore, the Fluent of this Series, *viz.* $x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\frac{1}{5}x^5+\frac{1}{6}x^6+\frac{1}{7}x^7+\frac{1}{8}x^8+\frac{1}{9}x^9+\frac{1}{10}x^{10}+\frac{1}{11}x^{11}+\frac{1}{12}x^{12}$ + &c. is = P*c*.

2°. Put P*p*=x, and *pa*=y; then $y=\frac{1}{1+x}$, and

and $y\dot{x} = \frac{\dot{x}}{1+x} =$ the Fluxion of Pa = (by throwing $\frac{\dot{x}}{1+x}$ into a Series) $\dot{x} - x\dot{x} + x^2\dot{x} - x^3\dot{x} + x^4\dot{x} - x^5\dot{x} + x^6\dot{x} - x^7\dot{x} + x^8\dot{x} - x^9\dot{x} + x^{10}\dot{x} - x^{11}\dot{x} +$ &c. Therefore, the Fluent of this Series, *viz.* $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \frac{1}{7}x^7 - \frac{1}{8}x^8 + \frac{1}{9}x^9 - \frac{1}{10}x^{10} + \frac{1}{11}x^{11} - \frac{1}{12}x^{12} +$ &c. is = Pa.

Hence,

If Pb=Pp, the Area $bcap$, viz. the Sum of Pc and Pa, will be $= 2 \times : x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \frac{1}{9}x^9 + \frac{1}{11}x^{11} +$ &c. And Pc—Pa will be $= x^2 + \frac{1}{2}x^4 + \frac{1}{3}x^6 + \frac{1}{4}x^8 + \frac{1}{5}x^{10} + \frac{1}{6}x^{12} +$ &c. That is, if Ob= .9, and bP=Pp=x= .1; then, by substituting .1 for x, we shall have ba = .2006706954 &c. and Pc — Pa= .0100503358 &c. half of which added to half ba, is .1053605156 &c.=Pc=the hyp. Log. of $\frac{1}{.9}$. And if Ob=.8, and bP=Pp =x= .2; then, by writing .2 for x, we have ba=.4054651081 &c. = the hyp. Log. of $\frac{1.2}{.8}$; and Pc—Pa=.0408219945 &c. half of which subtracted from half ba, is

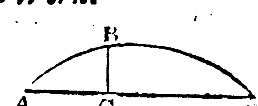
.1823215567 &c.

.1823215567 &c. $=Pa=$ the hyp. Log. of $\frac{1.2}{1}$ or 1.2; and this subtracted from ba, leaves .2231435513 &c. $=Pc=$ the hyp. Log. of $\frac{1}{.8}$. Therefore, by the Nature of Logarithms, the hyp. Log. of $\frac{1}{.9}\times\frac{1.2}{.8}\times 1.2$, *viz.* of 2, is $=$ .1053605156 &c. + .4054651081 &c. + .1823215567 &c. $=$ .6931471805 &c. And the hyp. Log. of 2^3, *viz.* of 8, is $=$ $3\times$6931471805 &c. $=$ 2.0794415416 &c. And the hyp. Log. of $8\times\frac{1}{.8}$, *viz.* of 10, is $=$ 2.0794415416 &c. + .2231435513 &c. $=$ 2.302585092 9 &c. *Q. E. I.*

[See *Stone's Math. Dict.* under the Word *Logarithms.*]

XIII.

In the Curve ABD, *whose Equation (putting* AC$=x$, CB$=y$, *and the given right Line* AD$=a$,) *is* $ax-x^2=ay+y^2$; *required the Radius of Curvature for any Point; and a geometrical Construction to illustrate and confirm the Work.*



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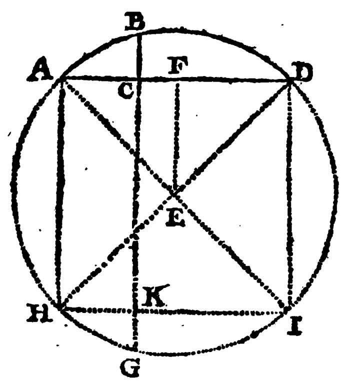
The *Fluxion* of the given Equation of the Curve, is $a\dot{x}-2x\dot{x}=a\dot{y}+2y\dot{y}$; which, making $\dot{x}=1$, is $a-2x=a\dot{y}+2y\dot{y}$; and the *Fluxion* of this Equation again, the Fluxion of $\dot{y}$ being considered as Negative, is $-2=-a\ddot{y}+2\dot{y}^2-2y\ddot{y}$. Hence we have $\dot{y}=\frac{a-2x}{a+2y}$, $\dot{y}^2=\frac{a^2-4ax+4x^2}{a^2+4ay+4y^2}$, and $\ddot{y}=\frac{4a^2+8ay+8y^2-8ax+8x^2}{\overline{a+2y}|^3}=$ (because by the Nature of the Curve, $8ay+8y^2=8ax-8x^2$, or $8ay+8y^2-8ax+8x^2=0$,) $\frac{4a^2}{\overline{a+2y}|^3}$. Now, by substituting for $\dot{y}^2$ and $\ddot{y}$ these their Values, in the general Expression for the Radius of Curvature, which was found in *Art.* 71. $=\frac{\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{\ddot{y}}$ when $\dot{x}=1$ and the Fluxion of $\dot{y}$ Negative; we shall have

$\overline{\frac{2a^2+4ay+4y^2-4ax+4x^2}{\overline{a+2y}|^2}}\Big|^{\frac{3}{2}}\times\frac{\overline{a+2y}|^3}{4a^2}=$ (because $4ay+4y^2-4ax+4x^2=0$,) $\overline{\frac{2a^2}{\overline{a+2y}|^2}}\Big|^{\frac{3}{2}}\times\frac{\overline{a+2y}|^3}{4a^2}=\frac{\overline{2a^2}|^{\frac{3}{2}}}{4a^2}=\frac{\overline{8}|^{\frac{1}{2}}a}{4}=\overline{\tfrac{1}{2}}|^{\frac{1}{2}}a=$ the Radius

dius of Curvature for any Point B required; which being a fixt or invariable Quantity, proves the Curve to be an Arch of a Circle.

Now, if the Radius of a Circle be $\overline{\tfrac{1}{2}}|^{\frac{1}{2}} a$; then, a is the Side of its inſcribed Square, or the Chord of 90°. as the right Line AD: For, if the ſaid right Line AD be biſſected in F,



and the Perpendicular FE drawn $=$ AF $=\tfrac{1}{2}a$; the Point E will fall in the Center of the Circle; and conſequently, the Radius will be AE $\left(=\overline{\overline{EF}|^2+\overline{FA}|^2}|^{\frac{1}{2}}\right)=\overline{\tfrac{1}{2}a^2}|^{\frac{1}{2}}=\overline{\tfrac{1}{2}}|^{\frac{1}{2}}a$. And, that x, in the given Equation, muſt flow in the ſaid Chord AD, may be thus demonſtrated. To any Point C, draw BC perpendicular to AD, and let it be produced till

it

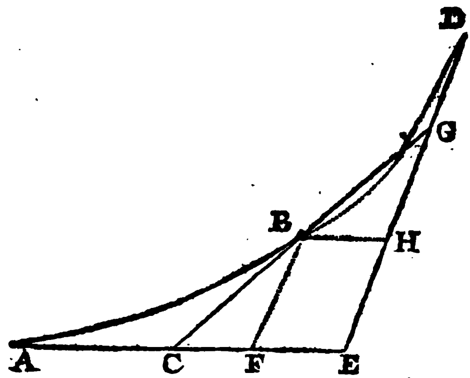
it meets the Circle's Periphery in G; and let HI be drawn parallel and equal to AD: then, 'tis evident, that, CK=AH=AD, by Conſtruction; and KG=CB, becauſe AC=HK, and AB=HG; therefore, CG=AD+CB: But, by a Property of Circles, AC×CD=BC×CG, *i.e.* (putting AD=a, AC=x, and CB=y,) $x \times \overline{a-x} = y \times \overline{a+y}$, or $ax-x^2 = ay+y^2$. Therefore, &c.

From the Conſtruction here given, it appears, that, the *Queſtion* may be infinitely diverſified, ſo as to be adapted to any regular Polygon, of an even Number of Sides, that can be inſcrib'd in a Circle.——We ſee here alſo, a Demonſtration of the Juſtneſs of the *fluxional Calculus*, as made Uſe of above.

XIV.

Let the given right Line CG, *in its firſt Situation, coincide with the right Line* AE; *and let the End* G *move along the right Line* ED *in ſuch a Manner, that the other End* C *may paſs on with an uniform Motion from* A *to* E; *and, at the ſame Time, ſuppoſe a Point* B *to move with the ſame uniform Motion, from* C *along the Line* CG: *Then, by the Motion of this Point, will the Curve*

ABD *be describ'd. Now,* BF *being always parallel to* DE, *'tis required to find the Point* C, *in the Line* AE, *when* CF *is a* Maximum.



PUT AE$=$ED$=$CG$=a$, AF$=x$, and AC$=$CB$=z$; then CF$=x-z$, BG$=a-z$, and FE$=$BH$=a-x$. By Sim. $\triangle s$, GB : BH : : BC : CF, *i.e.* $a-z : a-x : : z : x-z$; $\because ax-az-xz+z^2=az-xz$, or, $ax-2az+z^2=0$; which in *Fluxions* is $a\dot{x}-2a\dot{z}+2z\dot{z}=0$: But, when CF or $x-z$ is a *Maximum*, 'tis evident that $\dot{x}=\dot{z}$; therefore, by dividing by $\dot{x}$ or $\dot{z}$, we have $a-2a+2z=0$, or $2z=a$; and therefore $z=\frac{1}{2}a$, or, the Point C bissects the Line AE when CF is a Maximum. *Q. E. I.*

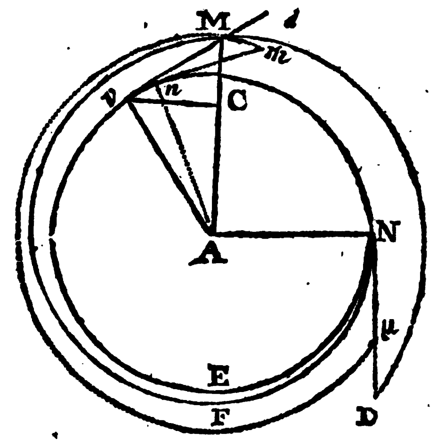
COROL-

Corollary. Since $CB = BG$; therefore, $CF = BH = FE = \frac{1}{2}CB$; and consequently, when the Angle AED is not an obtuse, but a right One, the $\angle BCF$ will, when CF is a *Maximum*, be $= 60^\circ$.

XV.

Let the given right Line AN *turn uniformly round the Point* A *as a Center; and, at the same Time, let a Point be suppos'd to pass with an uniform Motion, from* N, *along the right Line* ND, *equal and perpendicular to the said Line* AN, *and such Velocity, as to arrive at* D *at the same Time that the said Lines come to be in their first Situation: Then, by this Point, will the Spiral* NFMD *be describ'd. Quære the Area of any Space* $NE\nu MFN$?

Put $AN = ND = a$, Circle $NEN = b$, $\nu M = v$, Arch $\mu M = x$, $AM = y$, Arch $NE\nu = z$, $Mm = x'$, and $\nu n = z'$. Now, if we draw νC perpendicular to AM, the *Moment* or *Increment* of the Space $NE\nu M\mu$, *viz.* $\nu Mmn\nu$, will be $= \frac{1}{2} CM \times Mm =$ (because $AM : M\nu :: \nu M : MC$, or $y : v :: v : MC = \frac{v^2}{y}$,) $\frac{v^2 x'}{2y}$; therefore, the *Fluxion* of the said

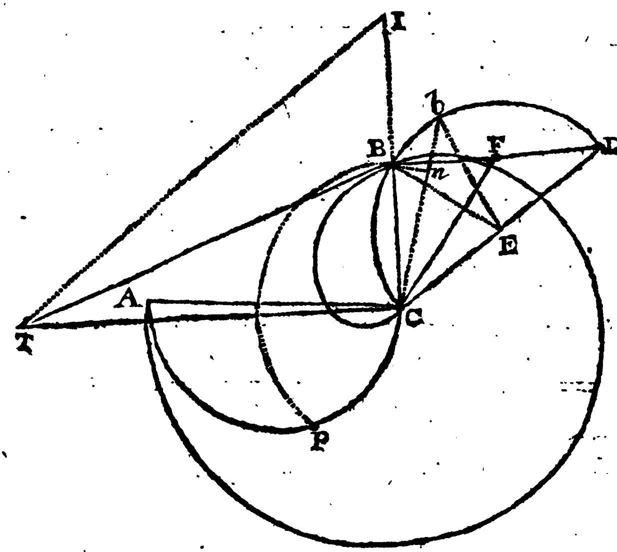


Space, which is = the *Fluxion* of the Space in *Question*, is $= \frac{v^2\dot{x}}{2y}$. But, it is plain from the Generation of the Curve, (vd) $a : b :: \dot{v} : \dot{z} = \frac{b\dot{v}}{a}$, and ($Av$) $a : y :: \dot{z} : \dot{x} = \frac{y\dot{z}}{a} = \frac{y}{a} \times \frac{b\dot{v}}{a} = \frac{by\dot{v}}{a^2}$; which ſubſtituted for $\dot{x}$, makes the above *Fluxion* of the Space in *Question* = $\frac{bv^2\dot{v}}{2a^2}$; the *Fluent* of which, is $\frac{bv^3}{6a^2}$ = the Area of the Space required. And, by writing a for v, we ſhall have the Area of the whole ſpiral Space NENDMFN $= \frac{1}{6}ab = \frac{1}{3}$ the Area of the Circle NEN. *Q. E. I.*

XVI.

XVI.

Let the Semicircle APC *turn uniformly round the Center* C; *in one Revolution of which, suppose a Point to move with an uniform Motion along the said Semicircle, from* C *to* A: *Then will this Point generate, or describe, the Spiral* CBA; *the Direction of which, at any Point* B, *is required.*



Let the Point b be supposed indefinitely near to B; and with the Ordinate CB, as a Radius, describe the Arch PBn. Put the Radius

Radius EB$=a$, Arch CB (of the generating Semicircle) $=v$, Arch PB $=x$, Ordinate CB$=y$; and B$b=v'$, $bn=y'$. Now, $a:2y::\dot{v}:\dot{x}=\frac{2y\dot{v}}{a}$; and, (becauſe by the Property of Circles $\angle b$EB$=2\angle$BCb,) $a:y::v':2$Bn, $\therefore$ B$n=\frac{yv'}{2a}$; but $\overline{Bb}|^2-\overline{bn}|^2=\overline{nB}|^2$, i. e. $v'^2-y'^2=\frac{y^2v'^2}{4a^2}$, or, $\dot{v}^2-\dot{y}^2=\frac{y^2\dot{v}^2}{4a^2}$; which Equation gives $\dot{v}=\frac{2a\dot{y}}{\sqrt{4a^2-y^2}}$. Hence we have $\dot{x}=\frac{2y}{a}\times\frac{2a\dot{y}}{\sqrt{4a^2-y^2}}=\frac{4y\dot{y}}{\sqrt{4a^2-y^2}}$; which ſubſtituted for $\dot{x}$, makes $\frac{\dot{x}y}{\dot{y}}$ (the general Expreſſion for the Subtangent CT, which is perpendicular to the Ordinate CB,) $=\frac{y}{\dot{y}}\times\frac{4y\dot{y}}{\sqrt{4a^2-y^2}}=\frac{4y^2}{\sqrt{4a^2-y^2}}=$ $\frac{2\text{BC}\times 2\text{BC}}{\text{BD}}=$ (by making BF $=$ FD,) $\frac{\text{BC}\times 2\text{BC}}{\text{BF}}$. Hence the following Conſtruction; *viz.* In the right Line CI, make IB$=$BC, and the $\angle$CIT$=\angle$CFB; then, T is the

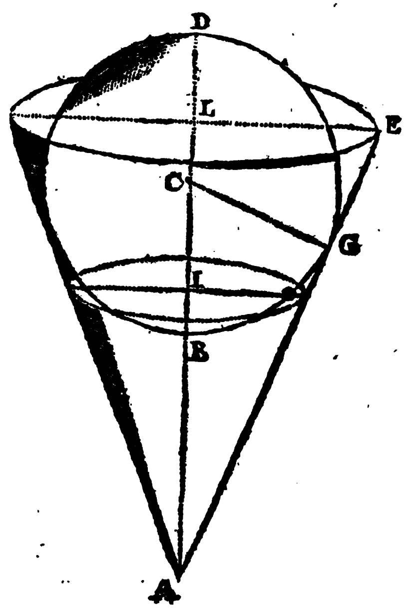
the Point from which the Tangent to the Point B muſt be drawn: For then, the right angled Triangles FBC and ICT will be ſimilar; and therefore, BF : BC : : IC : CT, *i.e.* BF : BC : : 2BC : CT $= \frac{BC \times 2BC}{BF}$; therefore, &c. *Q. E. I.*

COROL. Since $\dot{x} = \frac{2y\dot{v}}{a}$, and $Bn = \frac{y\dot{v}}{2a}$; therefore $\dot{x}$ or $x' = 4Bn$.

XVII.

If a heavy Sphere, whoſe Diameter is 4 Inches, be let fall into a conical Glaſs ⅕th full of Water, whoſe Diameter is 5 Inches and Altitude 6; how much of the Sphere will be immers'd in the Water?

PUT the Altitude $AL = 6 = a$; Radius $LE = 2.5 = b$; 3.14159 &c. $= c$; BD, the Diameter of the Sphere $= 4 = d$; and BI, that Part of the ſaid Diameter under the Water, $= x$; then, $ID = d - x$. Now, the Capacity of the Glaſs is $= \frac{1}{3} ab^2c$; and therefore, by *Queſt.* the Quantity of Water in it is $= \frac{1}{15} ab^2c$. By the Property of Circles, $BI \times ID = \overline{I\odot}|^2$, *i.e.* $dx - x^2 =$ the Square of the Radius



dius of the Section of the Sphere; therefore the Area of the faid Section is $= cdx - cx^2$; which drawn into $\dot{x}$, is $cdx\dot{x} - cx^2\dot{x} =$ the *Fluxion* of the Segment of the Sphere under the Water; the *Fluent* of which, is $\frac{1}{2}cdx^2 - \frac{1}{3}cx^3 =$ the Content of the faid Segment. Hence, (fimilar Solids being as the Cubes of their homologous Sides,) $\frac{1}{3}ab^2c : a^3 :: \frac{1}{13} \times ab^2c + \frac{1}{2}cdx^2 - \frac{1}{3}cx^3 : \frac{1}{3}a^3 + \frac{3a^2dx^2}{2b^2} - \frac{a^2x^3}{b^2}$

$=$

$=\overline{AI}|^3$. But, $AE=\sqrt{a^2+b^2}$; and $LE : EA :: GC : CA$, *i. e.* $b : \sqrt{a^2+b^2} :: \frac{1}{2} d : \frac{d}{2b}\sqrt{a^2+b^2} = CA$; $\because AB (=AC-BC) = \frac{d}{2b}\sqrt{a^2+b^2} - \frac{1}{2} d = 3.2$, which put $= e$; then $AI = e+x$, and $\overline{AI}|^3 = e^3+3e^2x+3ex^2+x^3 =$ (because by the above, AI^3 is $=$) $\frac{1}{5} a^3 + \frac{3a^2dx^2}{2b^2} - \frac{a^2x^3}{b^2}$. Now, this Equation produces $\overline{10a^2+10b^2}.x^3 + \overline{30b^2e-15a^2d}.x^2 + 30b^2e^2x = 2a^3b^2 - 10b^2e^3$, *i. e.* $422.5x^3 - 1560x^2+1920x = 652$; which divided by 422.5, is $x^3 - 3.692x^2+4.544x = 1.543$: Whence x may be found $= .546 = BI =$ that Part of the Diameter of the Sphere under the Water. *Q. E. I.*

FINIS.

ERRATA.

PAGE 6. *l.* 12. *f.* C*b*, *r.* *cb*. P. 24. *l.* 11, *f.* $=\frac{2}{3}$, *r.* $=-\frac{2}{3}$. P. 27. *l.* 7, *f.* 13, *r.* 12. P. 43. *l.* 13. *f.* y^2, *r.* z^2. P. 49. *l.* 9. *f.* by x, *r.* by $\dot{x}$. P. 73. *l.* 6. *f.* x, *r.* z. P. 79. *l.* 4, *f.* CBD, *r.* CBG. P. 90, *l.* 11, *f.* $x \times \overline{b^2-x^2}$, *r.* $x^4 \times \overline{b^2-x^2}|^{\frac{3}{2}}$. Ibid. *l.* 12, *f.* $x^4 . \overline{b^2-x^2}$, *r.* $x^4 . \overline{b^2-x^2}|^{\frac{3}{2}}$. P. 94, *l.* 7, *f.* 11, *r.* 10. P. 107, *l.* 8, *f.* $a\times$, *r.* $2\times$. P. 108, *l.* ult. *f.* y^{-2}, *r.* y^{m-2}. P. 153, *l.* penult, *f.* 4 . 6. *r.* 3.6. P. 165, *l.* 7, *f.* 2^2, *r.* a^2.

In ſome Copies are likewiſe the following *Errata*; *viz.* P. 129, *l.* 9, *f.* v, *r.* v^3. P. 158, *l.* 15, *f.* $\frac{1}{a+x|^2}$, *r.* $\frac{1}{\overline{a+x}|^2}$. P. 169, *l.* 8, *f.* $\overline{a^2+x^2}|^2$, *r.* $\overline{a^2+x^2}|^{\frac{1}{2}}$. P. 240, *l.* 11, *f.* $\overline{\frac{1}{2}a^2}|^2$, *r.* $\overline{\frac{1}{2}a^2}|^{\frac{1}{2}}$.

J. R. *July* 5, 1757.